Balancing Efficiency and Equity in Distribution Crises

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ABSTRACT

Distribution crises occur when there is a significant mismatch between the demand and supply of essential resources over an extended period. Without market regulations, these resources can be easily monopolized by the more affluent members of society, leaving the needs of the rest unfulfilled. While centralized planning can lead to fairer distribution, it also comes with economic and time-related inefficiencies. Following earlier literature on the topic, we propose to model the crises as multi-round trading environments that combine fair allocation of buying rights with a market-based approach, taking advantage of the benefits of both via rationing. The goal is to increase the flow of the resource to more buyers over time. However, while previous approaches were able to study such systems only empirically, we examine two implementations of the system's single-round markets and derive their exact analytical properties. The first market is a variation of the Arrow-Debreu model that incorporates buying rights, and we approximate its market-clearing solution through a polynomial-time auction algorithm. The second market involves non-myopic traders who optimize over multiple trading rounds. We determine one equilibrium of the second market and demonstrate that it is coalition-proof. To assess the entire environment's equity, we use the concept of frustration, defined as the scaled difference between the amount of goods a buyer is entitled to according to their buying rights and the amount they can acquire through trading. Our main results show that with both markets, the inclusion of buying rights reduces the expected per-round frustration by at least half.

KEYWORDS

Claims problem, Arrow-Debreu market, Fairness

1 INTRODUCTION

Crises of distribution arise when a vital resource becomes severely limited. Such crises can occur due to various reasons such as natural disasters, wars, or economic instabilities. Different mechanisms can be deployed to distribute resources in these situations. Two approaches may be considered as extremes: a centralized distribution by delegated authorities and a fully decentralized distribution through intervention-free markets. Markets have the potential to distribute goods flexibly and reliably among many buyers and sellers [4]. However, the free market has its drawbacks. In the absence of stable supply, scarcity can lead to significant price increases, with some traders attempting to acquire more than their "fair" share at the expense of others. This situation benefits the most powerful or well-connected individuals or organizations, leaving the less fortunate with limited or no access to the vital resources. On the other hand, centralized distribution by authorities can ensure a fairer division of resources based on pre-established rules. However, the resources are often not owned by the central authority. Furthermore, this approach comes with economic and time-related inefficiencies [9].

To combine the advantages of both approaches, in [11], the authors propose an iterative hybrid distribution system consisting of two stages. In each iteration, the central authority first distributes buying rights among the buyers in a desirable manner. Then, these rights are traded alongside the resource in a market. The rationale behind this design is to shift the market reallocation towards the desired centralized distribution, rather than allowing the distribution of scarce resources to concentrate among the wealthiest participants. However, the construction in [11] is very complex and focuses on immediate real-world deployability, rendering formal analysis unachievable. The authors hence resorted to empirical evaluation of the system, showing its efficiency in practice through reinforcement learning. In this work, we build on the same idea, but identify two mechanisms that allow us to explicitly derive the systems' (approximate) solutions and hence analyze the systems' efficiency analytically.

1.1 Structure of the proposed hybrid system

As in [11], our system consists of a sequence of interactions taking place iteratively over discrete time. For simplicity, we assume there is only one commodity to be distributed. In the market with myopic traders studied in section 3, this commodity consists of indivisible items, while in the market with non-myopic traders studied in section 4, the commodity is divisible. However, in contrast to the approach in [11], we suggest to frame the rights distribution as a resource allocation problem, which opens the door to leveraging the methods developed therein. Each interaction, referred to as the "extended market" among buyers and sellers, has two stages:

(1) The first stage exemplifies a "claim problem" as studied in axiomatic resource allocation literature [13]. In this stage, a finite amount of the critical commodity is divided among a set of buyers whose claims exceed

the available amount. Unlike traditional interpretations where conflicting claims arise from legal debt, in our case, claims represent buyers' "needs" for the critical resource. The authority divides the amount of the resource into buying rights of the same quantity, rather than directly into portions of the commodity, and the system moves to the second stage.

(2) The second stage is a market where two types of goods are traded through money: the critical commodity, only sold by sellers, and the buying rights, only sold by buyers. This stage can be implemented using various market models, but one crucial restriction must hold: at the end of each extended market, each buyer must own at least as many rights as the amount of the commodity they possess. All rights are eliminated at the end of each market, and the commodity acquired by buyers is consumed according to their needs.

To accentuate the desired centralized distribution perspective prioritized during crises, we refer to the division rules used in the first stage of the extended market to distribute the buying rights as the (rights) distribution mechanisms. In crises, these rules are often based on societal and ethical preferences and aim to divide the rights more equitably. Our goal is to minimize the role of the central authority while increasing equity compared to unregulated markets. The aim is to broaden the distribution of the essential commodity to buyers whom the central authority intends to allocate goods to but are unable to do so because of their limited financial resources. This is achieved by enabling buyers to accumulate funds through the sale of their buying rights during trading rounds.

To evaluate the effectiveness of this iterative system, we use the same measure as in [11] called "frustration". Frustration is defined as a normalized difference between the amount of rights assigned to a buyer and the actual amount of the commodity purchased by the buyer. This measure captures the degree to which buyers' claims are not fully satisfied. We aim to minimize frustration and achieve a more desirable distribution of resources.

1.2 Organization and contributions

Our primary contribution is the development of a formal system based on the autonomous behavior of traders, where solutions can be studied analytically, not empirically as in [11]. This system aims to shift the redistribution towards outcomes more preferable by the central authority during a distribution crisis, where there is a relatively stable but limited supply and excessive demand. The system operates through a sequence of trading interactions over discrete time, modeled as extended markets that distinguish the centralized allocation of buying rights and the trading of goods.

In Section 2, we formalize the hybrid trading environment, including the rights distribution mechanism and the measure of distribution effectiveness called frustration. These definitions are inspired by [11]. The rest of the paper consists of our original work. We analyze the evolution of the expected per-round frustration in this environment using two different implementations of single-round markets. To the best of our knowledge, we are the first to rigorously analyze a system that explicitly combines a market mechanism with a distribution mechanism to achieve a more desirable redistribution of critical goods during times of need. It is important to note that our findings are independent of the specific distribution mechanism used.

In Section 3, we delve into the analysis of solutions in systems where traders are myopic. We formulate an auction-based algorithm that can approximate the maximum clearing solutions of the extended market with myopic traders in polynomial time while providing a guarantee on its quality in Theorem 3.3. Using this result, we bound the expected frustration in the system in Theorem 3.5, showing that the equilibrium of the free market is moved towards the more desirable distribution.

Section 4 considers a more complex system with traders who optimize over the entire crisis. We examine properties of the resulting equilibrium and design an algorithm to compute it efficiently. Theorem 4.6 states the existence of an algorithm that computes a coalition-proof equilibrium of the system with non-myopic traders with linear utilities. Further analysis of the equilibrium reveals a similar asymptotic upper bound on the expected frustration as in the system with myopic traders, as stated in Theorem 4.8.

In the last part of the paper, we summarize the desired features of the trading system and provide several potential future directions. Due to space constraints, the full versions of all proofs and remarks are deferred to Appendices A and B.

1.3 Related work

The allocation of resources to individuals in a desirable – especially fair – manner has been extensively studied over the past decades. The objective of fair distribution is to identify an allocation mechanism that satisfies certain properties, commonly known as fairness criteria. In the literature, there is a wide variety of notions of fairness, and numerous works have explored the possibility of achieving both fairness and efficiency simultaneously. These works have examined fairness notions such as Envy Freeness, Pareto optimality, and maximum Nash welfare. A survey by [12] provides an overview of these notions. Our mechanism also draws extensively from the theory of claims and taxation problems, particularly fair divisions in bankruptcy problems, as surveyed in [13].

Our work contributes to the field of redistributive mechanisms, with a specific focus on reducing inequalities. In the literature, a related study examines a two-sided market for trading goods of homogeneous quality, aiming to optimize the total utility of traders [6]. However, our approach differs as we consider more general fairness measures beyond social welfare, and we assume that utilities are common knowledge. This line of research has been expanded to include settings with heterogeneous quality of tradable objects, various measures of allocation optimality, and imperfect observations of traders [1]. Another related work explores multiple market and non-market mechanisms for allocating a limited number of identical goods to multiple buyers [3]. The author argues that when buyers' willingness to pay aligns with the designer's allocation preferences, market mechanisms are optimal, and vice versa. In crises, where critical resources are highly valuable to all participants but some lack the necessary funds to acquire them, it is in society's best interest to allocate goods fairly. These findings suggest that relying solely on unregulated markets may not be the best approach during a distribution crisis.

Emissions allowances and tradable allowance markets share similarities with our work. Historically, regulators allocated tradable property rights directly to firms, leading to inefficiencies such as misallocation, regulatory distortions, and barriers to entry. Contemporary market designs utilize auctions for the allocation of tradable property rights. Tradable allowance markets, as discussed in [5], play a crucial role in ensuring the efficiency of carbon markets and preventing market power exertion by large and dominant agents. The efficiency of multi-round trading auctions for the allocation of carbon emission rights is studied in [14].

Arguably the most related work to ours studies a similarly constructed, yet more complex system, and studies its equilibria only empirically [11]. To the best of our knowledge, this is the first work that introduces such a hybrid trading-rationing mechanism and provides an analytical solution for such a complex rationing scheme.

2 PROBLEM STATEMENT

Formally, we model a crisis as a multi-round trading environment, which we call *Crisis*. We have one scarce resource, which we call *Good*. The other two commodities, representing funds and the right to purchase Good, will be referred to as *Money* and *Right*. Crisis consists of a sequence of $\mathcal{T}$ rounds. Each round is an extended market which we call Market. The traders in this trading environment form two disjoint sets of *sellers* and *buyers*. Sellers engage in selling goods, while buyers can participate in selling rights, buying rights, or purchasing goods. Each single-round Market of this multi-round trading environment starts with the sellers declaring the amount of Good for sale and the buyers declaring their *Claims* for Good. A central *rights distribution mechanism* then assigns buyers with appropriate amount of Right. At the beginning of each Market, each seller and buyer also receive an amount of Good and Money, respectively. A seller's declared quantity of Good equals their initial endowment of Good.

The traders then trade these *initial endowments* of Good, Right and Money: sellers are limited to selling Good and buyers are limited to buying Good and Right, selling Right and cannot sell Good. An important restriction is that *at the end of each Market, each buyer has at least as much Right as Good.* This restriction forms the core of our approach. We further require that Money obtained by selling Right cannot be used in the current Market for buying Good (see Remark A.1 for a justification). We refer to the single-round trading that happens after the Right is distributed as *Trading*. Hence, each Market consists of the distribution mechanism and the Trading.

At the end of each Market, (1) buyers consume all the obtained Right, (2) buyers consume all the obtained Good up to their declared Claim and keep the surplus Good and obtained Money to the next Market. (3) sellers only sell Good and consume obtained Money. Hence, the Markets of the sequence forming a Crisis are interdependent, see Appendix A.1.

2.1 Trading environment

Market number $\tau \leq \mathcal{T}$ is $\mathbb{M}^\tau = \mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$, where S is a set of sellers and B is a set of buyers. We denote $T = S \cup B$ the set of all traders and assume $S \cap B = \emptyset$. The $G^\tau = (G_t^\tau | t \in T)$ and $M^\tau = (M_t^\tau | t \in B)$ denote the sets of Good and Money each trader and buyer has at the beginning of a Market τ, respectively. We let $V^\tau \leq \sum_{s \in S} G_s^\tau$ be the offered volume of Good in the market $\mathbb{M}^\tau$. We also denote subsets of a set with a subscript, for example $G_A^\tau = (G_a^\tau | a \in A)$ for a set $A \subset T$. The set of Claim $D^\tau = (D_b^\tau | b \in B)$ gives the amount of Good each buyer hopes to acquire in the Market. The function u_t^τ is the utility of trader t in Market $\mathbb{M}^\tau$, defined as follows.

Definition 2.1 (Utility function of $\mathbb{M}^\tau$). The utility function of trader t in market $\mathbb{M}^\tau$ is $u_t^\tau : \mathbb{R}^2 \to \mathbb{R}$ where $u_t^\tau(x, y) = u_t^\tau(x, 0) + u_t^\tau(0, y)$ denotes the t's utility of amount x of Good and amount y of Money. We require that the utility function (1) is monotone in each coordinate. Moreover, for each trader t, (2) $u_t^\tau(0, x)$ depends linearly on x, (3) sellers have a positive utility only for Money, and (4) For each buyer b, $u_b^\tau(x_1, 0) + u_b^\tau(x_2, 0) \geq u_b^\tau(x_1 + x_2, 0)$.

Finally, ϕ^τ is the (rights) distribution mechanism of Market $\mathbb{M}^\tau$. It has the following form.

Definition 2.2. A distribution mechanism is a function $\phi : \mathbb{R}_0^{+,|B|+1} \to \mathbb{R}_0^{+,|B|}$ which, given the offered volume of Good V and Claim D of buyers, assigns allocation of Right to the buyers which satisfies $\forall V, V' \in \mathbb{R}_0^+, D \in \mathbb{R}_0^{+,|B|}, \forall b \in B, \forall \tau \in \mathbb{N}$ and all permutations of $|B|$ elements α

(1) $\sum_{b \in B} \phi_b(V, D) = V$,
(2) $D_b = 0 \Rightarrow \phi_b(V, D) = 0$,
(3) $D_b \geq D_b' \Rightarrow \phi_b(V, D) \geq \phi_b(V, D')$,
(4) $\phi_b(V, \alpha(D)) = \phi_{\alpha^{-1}(b)}(V, D)$,
(5) $V \geq V' \Rightarrow \phi_b(V, D) \geq \phi_b(V', D)$.

To work in the system as intended, the distribution mechanism has to distribute Right for all offered Good, assign no Right to buyers without Claim for the scarce resource, be non-decreasing with Claim of each buyer, and treat buyers with equal Claim equally. Our results do not depend on a particular choice of a distribution mechanism. We include two well known examples for illustration purpose only.

Example 2.3. One example of a distribution mechanism is the *proportional fairness mechanism*, which allocates Right to b proportionally to their Claim, i.e.,

$$\phi_b\left(\sum_{s \in S} v_s^\tau, D\right) = \frac{D_b \sum_{s \in S} v_s^\tau}{\sum_{b \in B} D_b}, \qquad \alpha_b^\phi(D, V) = \frac{D_b}{\sum_{b \in B} D_b},$$

where the distribution is not dependent on the total volume distributed. Another example of a distribution mechanism is the *contested garment distribution* [2] designed to fairly resolve conflicting claims.

Crisis

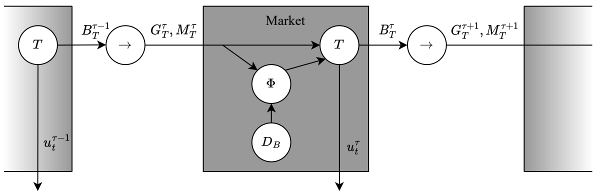

Figure 1: An illustration of a Crisis consisting of a series of Markets. The distribution function ϕ is followed by the Trading phase, T. The baskets obtained by traders are transferred to the next Market.

Assigning the $\phi_b(V^\tau, D^\tau)$ amount of Right to each buyer b by the distribution mechanism ϕ constitutes the first of the two steps of a Market number τ, $\mathbb{M}^\tau$. In the second step, traders trade assigned Good, Right and Money in the Trading. The Trading is a standard market with two restrictions: (1) the final basket of each buyer has the amount of Right at least as big as the amount of Good and (2) Money obtained for selling Right cannot be used to buy Good in the current Trading.

Definition 2.4 (Solution of $\mathbb{M}^\tau$). Let X be a set containing amounts of Good, Right and Money. We use $\mathcal{M}(X)$ ($\mathcal{G}(X), \mathcal{R}(X)$ respectively) to denote the amount of Money (Good, Right respectively) in X. A *solution* of Market $\mathbb{M}^\tau$ consists of (1) the price q^τ (p^τ respectively) per unit of Right (Good respectively), and (2) a partition of a subset of the union of all the initial endowments into *baskets* B_t^τ, $t \in T$. Given a solution, we say that the assigned basket B_t^τ of a trader t is *t-feasible* if it satisfies, assuming the price of a unit of Money is 1, that the total price of B_t^τ is at most the total price of t's initial endowment, and further if t is a buyer then (1) $\mathcal{R}(B_t^\tau) \geq \mathcal{G}(B_t^\tau)$ and (2) B_t^τ contains the amount of Money which t obtained from selling Right, i.e., $\mathcal{M}(B_t^\tau) = M_t^\tau - p^\tau \mathcal{G}(B_t^\tau) + q^\tau(\phi_t^\tau(V^\tau, D^\tau) - \mathcal{R}(B_t^\tau))$. A solution is *feasible* if for each trader t, B_t^τ is t-feasible.

As remarked earlier, the subsequent Markets are not independent; a feasible solution of $\mathbb{M}^\tau$ influences initial endowments of Money in $\mathbb{M}^{\tau+1}$. The cascaded Markets formally define the *Crisis* as a sequence $\mathbb{C} = \mathbb{C}(S, B, \mathcal{T}, G, M, D, u, \phi, \mathcal{B})$ of $\mathcal{T}$ Markets and *their feasible solutions*, where S and B have the same meaning as in the previous subsection. Further, $G = (G^\tau : \tau \leq \mathcal{T})$, $M = (M^\tau : \tau \leq \mathcal{T})$, $D = (D^\tau : \tau \leq \mathcal{T})$, $u = (u^\tau : \tau \leq \mathcal{T})$ and $\phi = (\phi^\tau : \tau \leq \mathcal{T})$. Finally, $\mathcal{B} = (B_t^\tau : \tau \leq \mathcal{T}, t \in T)$ denotes the vector of all baskets across traders and market iterations and each $\mathcal{B}^\tau = (q^\tau, p^\tau, (B_t^\tau; t \in T))$ is a feasible solution of Market $\mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$. We further require that for each $\tau < \mathcal{T}$ and $t \in B$, $M_t^{\tau+1} \geq \mathcal{M}(B_t^\tau)$ (see Remark A.1). An illustration of the structure of the Crisis is available in Figure 1.

2.2 Frustration per-buyer and in expectation

The concept of the Right can be understood as the socially determined entitlement of a buyer to a specific amount of Good. Put simply, if the distributional crisis were entirely managed by a central authority, the buyer would receive precisely that amount. The concept of "frustration" aptly captures the contrast between this ideal centralized solution and the actual quantity a buyer is able to obtain.

Formally, the frustration of a buyer is the normalized difference between the Right he would be assigned and the amount of Good he acquired in a Market if that is at least zero, and zero otherwise. We note that this definition allows measuring frustration also in the free market, where no Right exists. However, a common notion of desirable allocation may still exist. Minimization of frustration captures the objective of moving towards the desirable distribution.

Definition 2.5. Let $\mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$ be a Market. Then the *frustration* of buyer $b \in B$ of a non-zero Claim in market $\mathbb{M}^\tau$, denoted by f_b^τ, is

$$f_b^\tau = \max\left\{0, \frac{\phi_t^\tau(V^\tau, D) - \mathcal{G}(B_t^\tau)}{\phi_t^\tau(V^\tau, D)}\right\},$$

When the Good is traded, the final allocation may differ from the centralized distribution. This disparity serves as a measure of the inefficiency inherent in trading when it comes to allocating the Good in a manner that aligns with the central authority's preferences. This concept bears resemblance to the Price of Anarchy, which quantifies the cost incurred by the system due to the autonomous behavior of the involved actors [8]. We define the expected frustration in our system as a scaled sum of frustrations of the buyers, i.e.,

$$\mathbb{E}_f^\tau = \frac{\sum_{i=1}^{\tau} \sum_{b \in B} f_b^i}{\tau |B|}.$$

Note that $\mathbb{E}_f^\tau \geq 0$ since $f_b^\tau \geq 0$ and $\mathbb{E}_f^\tau = 0 \Leftrightarrow f_b^i = 0\ \forall b \in B, \forall i \in \{1, \ldots \tau\}$. In the latter case, trading Good in the Market has the same social impact as distributing it centrally.

3 MARKET WITH MYOPIC TRADERS

In this section, we study myopic traders who always optimize the current Market, i.e., we measure the optimality of the feasible solution of a multi-round trading environment by how optimal the feasible solutions of each round are. In this section, we assume the Good is indivisible. We begin by defining a concept of an optimal solution of Market $\mathbb{M}^\tau$.

Definition 3.1 (Optimal solution of $\mathbb{M}^\tau$ and $\mathbb{C}$). A solution of the Market is *optimal* if for each trader t, (1) B_t^τ is t-feasible and (2) $u_t^\tau(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau))$ is maximum among all $u_t^\tau(\mathcal{G}(Q), \mathcal{M}(Q))$, where Q is a t-feasible subset of the union of all the initial endowments. We further say a solution of a Crisis is optimal if for each τ, $\mathcal{B}^\tau$ is an optimal solution of $\mathbb{M}^\tau$.

3.1 Auction-based approximation algorithm

We introduce an efficient auction-based algorithm for finding an approximate optimal solution of Market $\mathbb{M}^\tau$. The algorithm is conceptually similar to the standard approximation algorithm for the Arrow-Debreu model [10]. Since our discussion is valid for any Market, we will omit the upper index τ in this section to ease notation. We will assume in this section that Good (and thus also Right and Money) are *indivisible*. This can be more practical for distribution crises since commodities often come in packages. Analogous results with simpler proofs hold also for divisible commodities.

For the purpose of the algorithm, we introduce a new commodity called *Couple* as a pair (x_1, x_2) where x_1 is an item of Good and x_2 is an item of Right. For a trader t, we will assume their *utility of x items of Couple* is the same as of x items of Good, i.e., equal to $u_t(x, 0)$. The algorithm auctions items of Couple. We will denote the current price of one item of Good (Right, Couple respectively) by p (q, c respectively). We assume that the price of one item of Money is equal to 1. Finally, we recall that M_t denotes the initial endowment of Money of trader t and for ease of notation, we denote by R_t the initial endowment of Right of t.

The algorithm description. Let $0 < \epsilon < 1$. The algorithm is divided into *iterations*. During each iteration, some items of Couple are sold for c and some for $(1+\epsilon)c$, and analogously for Good and Right. Each iteration is divided into *rounds*. An iteration ends when the price of Couple is raised from c to $(1+\epsilon)c$. Initially, we let $p = q \leftarrow 1, c \leftarrow 2$ and each buyer gets the surplus *cash* covering its initial endowment of Money and Right. Cash is a dummy commodity representing the flow of Money in the system. The Algorithm finds traders' output baskets and establishes the price. Only the output of the Algorithm needs to adhere to the rules of a Market. The application of the Algorithm is restricted to situations where no buyer possesses a positive surplus of Good in a Market, as indicated by Definition 3.2(3).

Round: we fix an arbitrary order of buyers and consider them one by one in this order. Let buyer b be considered. Let us denote by o^b the number of items of Couple b currently has, and by o^b_+ the number of items of Couple b currently has of price $(1+\epsilon)c$.

Let S^b be a set of items of Couple and of Money of max total utility which b can buy with its current cash plus co^b. Let $C(X)$ denote the number of items of Couple in set X. If $C(S^b) < o^b$ then b does nothing, the algorithm moves to the next buyer[1]. If $C(S^b) \geq o^b$ then b buys items of Couple via the **Outbid**:

- We keep as the invariant of the algorithm that the cash of each buyer b is always at least $q(R_b - C(B_b))$ where B_b is b's current basket.
- The system buys with cash one by one and at most $C(S^b) - o^b_+$ items of Couple for price c and sells them to b for cash price $(1+\epsilon)c$ per item, maintaining the invariant. First, it buys from b itself. We remark that, when the system purchases a Couple (x_1, x_2) for buyer b, the x_1 and x_2 may originate from different traders.
- An alternative for buying the items of Couple is to buy separately items of Good and Right, possibly from different traders, and compose them into items of Couple. This happens when some items of Right and (necessarily the same amount of items of) Good are not yet coupled in previous tradings. We observe that this happens only if they are available for the initial price from the traders. In this situation, the system again buys items of Right first from the buyer b. However, the system pays nothing if it buys items of Right from an initial endowment of a buyer for the initial price since the payment is already in the surplus cash.
  - If no more Couple is available at price c after the Outbid then the current round and iteration terminate, $p \leftarrow (1+\epsilon)p$, $q \leftarrow (1+\epsilon)q$, $c \leftarrow (1+\epsilon)c$ and the cash is updated: everybody who had Good or Right in its initial endowment gets extra cash, ϵp per item of Good or ϵq per item of Right.
  - If a round went through all buyers, the algorithm proceeds with the next round.
  - When nobody wants to buy new items of Couple, the whole trading ends. The system takes all items of Money from the buyers, sells them to the buyers and sellers for cash and keeps whatever items remain.
  - The output of the algorithm consists of (1) the collection of the final baskets of each trader and (2) the terminal prices p, q, c.

[1] If this happens then the current basket of b is optimal for the previous price $c/(1+\epsilon)$ and $o^b_+ = 0$.

We are able to analyze the algorithm only when initial endowments are feasible.

Definition 3.2. Let $0 < \epsilon < 1$. The initial endowments $(R_b, M_b); b \in B$ are *feasible for* ϵ if they satisfy, for each buyer $b \in B$, the following properties: (1) $M_b > \max(2/\epsilon, 4R_b)$, (2) for each $x \leq R_b$, $u_b(x, 0) \geq 2u_b(0, x)$ and (3) for each $x \geq \min(D_b+1, 1/2M_b), u_b(0, M_b) \geq u_b(0, M_b-x)+u_b(x, 0)$. Moreover, the offered volume of Good is *feasible* if $V^\tau = \sum_{s\in S} G^\tau_s$.

The concept of feasibility of initial endowments hinges on limiting the quantity of goods purchased not by the available money but by utility (subadditive for goods and additive for money, as defined in Definition 2.1). Specific constants are established to ensure the validity of Theorem 3.3. The feasibility assumptions are not restrictive for applications where the buyers are institutions (e.g., hospitals) which have practically unlimited amount of money and use their individual utility function to decide how to spend them. The assumption that a buyer b has no utility from Good exceeding his Claim is natural, especially in the middle of a distribution crisis. This makes b buy at most his Claim of Good in every Market. Note it is also favourable for the central authority, as it possesses comprehensive information about the trading Rights and can verify them efficiently. Finally, the feasibility of the offered volume of Good makes sense for *myopic* sellers.

Theorem 3.3. *Let $0 < \epsilon < 1$. We assume that initial endowments of buyers are feasible for ϵ and the offered volume of Good is also feasible. We further assume that each buyer in every round does one algorithmic step. The following holds.*

1. *The time-complexity of the auction-based algorithm is at most $|B|^2 \left(1 + \log_{1+\epsilon} \sum_{b\in B} M_b\right)$; hence, the auction-based algorithm is polynomial in the input size.*
2. *For each participant, the basket assigned by the algorithm is feasible and its price plus 1 is bigger than the total price of its initial endowment.*
3. *The terminal price of Right is equal to the terminal price of Good.*

(4) *Relative to terminating prices, each buyer or seller gets a basket of utility at least* $(1-2\epsilon)$ *times the utility of its optimal feasible basket.*

See also Appendix A.2.

3.2 Upper-bounding the frustration

In order to make observations on the developing frustration of Markets with myopic traders, we make several restrictive assumptions.

Definition 3.4. We say a Crisis is measurable if

(1) The total supply and the individual demand do not change in the sequence of Markets.
(2) The individual utility of Good in each $\mathbb{M}^\tau$ is such that each buyer b wants to spend all available Money for buying Good, no matter its amount and price.
(3) The amount of Money each buyer b is willing to spend for buying Good is non-decreasing. Specifically, let us denote by m_b^τ the amount of Money b is willing to spend in the Market $\mathbb{M}^\tau$ and by z_b^τ the amount of Money b he obtained by selling Right in $\mathbb{M}^\tau$. For each τ,

$$m_b^{\tau+1} = m_b^\tau + z_b^\tau.$$

The measurability assumptions are quite restrictive but arguably natural in our regime. In this paper, we study only the regime when the Markets happen deep in a distribution crisis when Good is scarce. Furthermore, in this section, the buyers are assumed to have practically unlimited funds and only care about how much they want to allocate towards buying Good. See comments after Definition 3.2.

THEOREM 3.5. *In all but possibly the first Market of a measurable Crisis where trading is implemented by the auction-based algorithm of section 3.1, each individual frustration is at most* $1/2$.

4 MARKET WITH NON-MYOPIC TRADERS

While the previous section considered market-clearing as an optimal solution of a system with myopic traders and the Good was indivisible, this section investigates solutions of multi-round trading systems with traders capable of optimizing over a long horizon, and the Good is divisible. We derive an explicit formulation of the solution in the form of the interaction's equilibrium, which turns out to be a simple extension to the free market's equilibrium. Furthermore, we examine its robustness with respect to coalitions, and formulate an upper bound on the arising expected frustration.

4.1 Game-theoretic formulation

Considering non-myopic traders substantially increase the complexity of the situation, making the Crisis a sequential game. Let us first introduce a specific extension of the single-round Market with such traders that we call the *Market game.*

Formally, the Market game is a tuple $(\mathbb{M}^\tau, \Pi, \mu)$, where $\mathbb{M}^\tau$ is a Market as defined previously, with particular utilities for sellers and buyers. The sellers are motivated by profit, so their utility is the amount of Money they acquire in a Market. We modify this simple model by subtracting the amount of Good they are left with after the Market. This represents the cost of storing, as well as the damaged reputation by not selling the scarce Good. Together,

$$u_s^\tau(\mathcal{G}(B_s^\tau), \mathcal{M}(B_s^\tau)) = \mathcal{M}(B_s^\tau) - c\,\mathcal{G}(B_s^\tau),$$

where $c > 0$ is a suitable constant. A buyer, on the other hand, wishes to keep a steady supply of Good during the Crisis. Here, we assume each buyer makes use only of the Good they consume at the end of the Market via

$$u_b^\tau(\mathcal{G}(B_b^\tau), \mathcal{M}(B_b^\tau)) = \min\left\{D_b, \mathcal{G}(B_b^\tau)\right\}.$$

We denote by Π the set of strategies of the traders. We assume each seller has information about the amount of Good every seller has, as well as Money and Claim[2] each buyer has. On the other hand, each buyer has access only to the amount of Money and Good they already own, as well as their Claim. We consider this assumption sufficiently realistic as the sellers may invest in some market research, while the availability of such information to the buyers remains rather limited. The strategy of each seller is thus a function $\pi_s : \mathbb{R}_0^{+,2|B|+|S|} \to \mathbb{R}_0^{+,2}$, denoted as

$$\pi_s(G_B^\tau, M_B^\tau, G_S^\tau) = (v_s^\tau, p_s^\tau),$$

where $v_s^\tau \leq G_s^\tau$ is the offered volume of Good at price p_s^τ by seller s in Market τ. Based on the total offered volume of Good $V^\tau = \sum_{s\in S} v_s^\tau$ and Claim of buyers, the distribution mechanism ϕ allocates the Right. When offering Right for sale, a buyer is given the offers of sellers and the amount of Good, Money and Right they have. Their strategy for this task is hence a function $\hat{\pi}_b : \mathbb{R}_0^{+,2|S|+3} \to \mathbb{R}_0^{+,2}$, denoted as

$$\hat{\pi}_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau) = (w_b^\tau, q_b^\tau),$$

where $w_b^\tau \leq R_b^\tau$ is the offered volume of Right at price q_b^τ. When declaring acceptable price and volume, a buyer is also given the offers of the other buyers. Summarizing, the (complete) strategy of a buyer b is a function $\pi_b : \mathbb{R}_0^{+,2|S|+3+2(|B|-1)} \to \mathbb{R}_0^{+,6}$, written as

$$\pi_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau, w_{-b}^\tau, q_{-b}^\tau) = (w_b^\tau, q_b^\tau, \overline{v}_b^\tau, \overline{p}_b^\tau, \overline{w}_b^\tau, \overline{q}_b^\tau),$$

where $\cdot_{-b}^\tau = \{\cdot_{b'}^\tau | b' \in B \setminus \{b\}\}$ and $\bar{\cdot}_b^\tau$ denote acceptable amounts of the corresponding quantities for buyer b. We will denote $\pi = \pi_S \times \pi_B$ the strategy profile of all traders. After all traders declare their bids, the Trading begins, which is done via our particular market mechanism.

Definition 4.1. (informal [3]) The market mechanism is a function $\mu : \Pi \times \mathbb{R}_0^{+,2|T|+|B|} \to \mathbb{R}_0^{+,2|T|}$, written as

$$\mu(v_S^\tau, p_S^\tau, w_B^\tau, q_B^\tau, \overline{v}_B^\tau, \overline{p}_B^\tau, \overline{w}_B^\tau, \overline{q}_B^\tau, G_T^\tau, M_T^\tau, R_B^\tau) = B_T^\tau,$$

where B_t^τ is the basket containing the amount of Good and Money t gained during trading. The market mechanism we consider has two stages. In the first stage, the buyers use the Right they were assigned to buy as much Good as they desire. In the second stage, the buyers buy Good and Right in equal volume, until they buy their desired volume of either, or they

[2] Since the distribution mechanism is assumed to be public knowledge, the traders also know (for some offered volume of Good) the amount of Right each buyer will be assigned.

[3] See Appendix C for the formal definition.

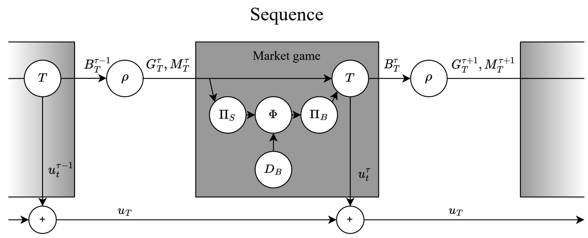

Figure 2: An illustration of the Sequence of Market games, see main text.

have no Money left. In both stages, items offered at a lower price are traded first. When more traders offer Good or Right at the same price, they are treated as a single trader until one runs out of items for sale.

Similarly as in the previous section, we are interested in the performance of the system over the period of multiple rounds, each realized by a Market game. To distinguish this implementation of Crisis with non-myopic traders from its myopic alternative, we refer to the sequential game consisting of $\mathcal{T}$ Market games as the *Sequence*. Formally, the Sequence is a tuple $\mathbb{S}(\mathcal{T}) = (S, B, D, u, \mathcal{T}, \phi, \mu, \omega, \rho)$, where S, B, D, ϕ and μ have the same meaning as before, and ρ is a *transition* function. The purpose of the transition function is to consume resources[4] after each Market, and give traders additional Good and Money for the next Market.

Definition 4.2. The transition function $\rho : \mathbb{R}_0^{+,2} \to \mathbb{R}_0^{+,2}$ for a seller $s \in S$ and a buyer $b \in B$ is given by

$$\rho(\mathcal{G}(B_s^\tau), \mathcal{M}(B_s^\tau)) = (g_s + \mathcal{G}(B_s^\tau), 0)$$
$$\rho(\mathcal{G}(B_b^\tau), \mathcal{M}(B_b^\tau)), = (\max(0, \mathcal{G}(B_b^\tau) - D_b), m_b + \mathcal{M}(B_b^\tau)),$$

where g_s and m_b are the amount of Good and Money sellers and buyers gain after each Market game respectively.

Our results rely on the assumption that the quantities g_s and m_b are constant during the Sequence. Without a loss of generality, we set $\sum_{s \in S} g_s = 1$ and $\sum_{b \in B} m_b = 1$. This assumption is not only crucial for our analysis but also practical in the midst of a crisis.

The structure of the entire Sequence is visualized in Figure 2, see also Algorithm 3 in Appendix C for more details. We further assume the traders aggregate their utilities over the whole Sequence so that their resulting outcomes are $u_t = \sum_{\tau=1}^{\mathcal{T}} u_t^\tau$. As a solution of this interaction, we employ the standard notion of equilibrial strategies, such that no deviation of a trader may increase their utility.

Definition 4.3. For a Sequence $\mathbb{S}(\mathcal{T})$, a strategy $\pi \in \Pi$ is an equilibrium, if for any other strategy profile $\overline{\pi}_t$ of any $t \in T$

$$\sum_{\tau=1}^{\mathcal{T}} u_t^\tau(\pi) \geq \sum_{\tau=1}^{\mathcal{T}} u_t^\tau(\overline{\pi}_t, \pi_{-t}), \qquad \forall t \in T,$$

where $u_t^\tau(\pi)$ is the utility received by trader t in Market τ under strategy profile π.

[4] Formally, we also need to remove Money of sellers to prevent them from obtaining utility for it multiple times.

4.2 Greedy strategy

At first glance, it may be unclear how the inclusion of Right in trading affects the strategies of rational traders. We will demonstrate that a particular equilibrium of the Sequence is a natural extension of the free market equilibrium, with a strictly lower expected frustration. Informally, this proposed strategy is that the sellers post the highest price that the buyers can afford to pay, taking into account the cost of Right. As a result, buyers will purchase all available Good, which also means some will purchase Right. Those who sell Right will do so for the same price as for the Good. Buyers will not accept prices higher than this for either Right or the Good. Finally, sellers will not offer more Good than they receive at the beginning of each Market. We refer to such strategies as *Greedy*, and they will be of our focus further on.

Definition 4.4. The Greedy strategies of the sellers are

$$\pi_s(G_B^\tau, M_B^\tau, G_S^\tau) = (g_s, p^\tau), \tag{1}$$

where p^τ is the solution of

$$\sum_{b \in B} M_b^\tau - \max(0, p^\tau R_b^\tau - M_b^\tau) = p^\tau \sum_{b \in B} R_b^\tau. \tag{2}$$

For buyers, the Greedy strategies $\pi_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau, w_{-b}^\tau, q_{-b}^\tau)$ are

$$\left(\max(0, R_b^\tau - M_b^\tau/P^\tau), P^\tau, R_b^\tau, P^\tau, \max(0, M_b^\tau/P^\tau - R_b^\tau), P^\tau\right), \tag{3}$$

where $P^\tau = \frac{\sum_{s \in S} p_s^\tau}{|S|}$ is the average selling price of Good.

We justify the exact form of Eq. (2) in the next section, see also Appendix A.3.

Our primary focus is on a shortage of Good, which we formalize by demanding that each buyer can never purchase more than their Claim, even if they try to buy as much as possible in every Market of the Sequence. This is analogous to point (3) of Definition 3.2. To be more specific,

Definition 4.5. Let $\mathbb{M}^\tau$ and $\mathbb{S}$ be a Market and a Sequence respectively. We say $\mathbb{M}^\tau$ has a feasible initial endowment, if, assuming traders utilize the Greedy strategy outlined in Definition 4.4, then $D_b > \mathcal{G}(B_b^\tau)$, $\forall b \in B$. We say $\mathbb{S}$ is feasible if every Market of $\mathbb{S}$ is feasible.

4.3 Equilibrium of the Sequence

We will now demonstrate how the Market operates when all traders follow the Greedy strategy. All sellers and buyers post the same selling prices, p^τ and q^τ, respectively, for the Good and Right. The amount of Good that all sellers can sell in the first stage is $\sum_{b \in B} \min\{R_b^\tau, M_b^\tau/p^\tau\}$. We will call a buyer *rich* if $p^\tau R_b^\tau < M_b^\tau$, and *poor* otherwise. In the second stage, poor buyers will offer all the Right that they cannot use in the first stage, as they know the prices set by sellers before offering Right. The amount of offered Good is $\sum_{b \in B} R_b^\tau$, which means the amount of Good sold in the second stage is

$$\sum_{b \in B} R_b^\tau - \min(R_b^\tau, M_b^\tau/p^\tau) = \sum_{b \in B} \max(0, R_b^\tau - M_b^\tau/p^\tau).$$

The amount of Money sellers get from both stages is

$$\Delta M_S^\tau = \sum_{b\in B} p^\tau R_b^\tau,$$

since all Good is sold, see Appendix A.3. Note that this is less than or equal to the amount of Money the buyers posses in total because the Money used for selling Right may only be used in the following Market. That Money is the amount of Right sold in the second stage times the price

$$\Delta M_B^\tau = \sum_{b\in B} q^\tau \max(0, R_b^\tau - M_b^\tau/p^\tau) \geq 0, \tag{4}$$

where the equality holds only when $p^\tau R_b^\tau = M_b^\tau, \forall b \in B$, so they can all buy everything in the first stage. The inequality of the buyers thus dictates how much of the Money can be used to buy Good. We will refer to ΔM_S as *useful Money* in a Market and to ΔM_B as *useless Money*. The total Money in the system is the sum of both, i.e.,

$$\sum_{b\in B} M_b^\tau = \Delta M_S^\tau + \Delta M_B^\tau = \sum_{b\in B} p^\tau R_b^\tau + q^\tau \max(0, R_b^\tau - M_b^\tau/p^\tau).$$

We can also derive this equation by taking into account that the cost of the Good is p^τ in the initial stage and $p^\tau + q^\tau$ in the subsequent stage. This implies that

$$\sum_{b\in B} M_b^\tau - \frac{q^\tau}{p^\tau} \max(0, p^\tau R_b^\tau - M_b^\tau) = p^\tau \sum_{b\in B} R_b^\tau, \tag{5}$$

which offers a nice intuition: the Money each buyer has is effectively decreased by the (scaled) frustration which p^τ would induce. This is because the rich buyers will buy the amount of Right exactly equal to the frustration of the poor, hence the scaling by the relative price. If the traders follow Greedy, then $q^\tau = p^\tau$ and Eq. (5) reduces to Eq. (2), see Appendix A.3.

Theorem 4.6. *The Greedy strategies given in Eq.* (1) *and* (3) *form an equilibrium of a feasible Sequence of any length. Furthermore, the equilibrium is coalition-proof and can be computed efficiently.*

When players follow the Greedy strategy, all Goods are sold, and the amount of Goods offered remains the same in every Market. Therefore, we will no longer include the upper index of allocated Right in the rest of this section and will use R_b to denote Right allocated to b in any Market. It is also worth noting that, under these assumptions, $\sum_{b\in B} R_b = 1$.

The existence of an analytic solution for a Sequence with an arbitrary distribution mechanism remains uncertain. However, in certain specific cases, a solution can be found. One such example is a system with only one buyer, which is akin to a free market. Additionally, if the ratio R_b/m_b is the same for all buyers, the system again reduces to a free market, as can be observed from Eq. (4). Notably, an analytic solution also exists if all the rights are allocated to a single buyer, which implies a more general result, see Appendix D.

4.4 Upper-bounding the frustration

One may observe certain similarities between Eq. (5) and (6) – the frustration decreases useful Money a buyer has, and increases the amount they will have in the following Market. Now we will show that the price oscillates around a fixed value and tends to it over time. Moreover, that fixed value is the (free-)market clearing price.

Proposition 4.7. *Let all traders follow the Greedy strategy. Then the mapping of the current price to the next one is a non-expansive mapping on $\mathbb{R}$ with the L1 norm, resulting in the limiting price being one.*

This suggests that the Sequence ultimately reaches a stable state. Additionally, the price is the same as it would be in a free market $\frac{\sum_{b\in B} m_b}{\sum_{s\in S} g_s} = 1$. This means the distribution mechanism primarily impacts the income of buyers.

Proposition 4.7 illustrates that if all traders employ Greedy strategies, the Sequence eventually stabilizes. In this scenario, the amount of Money and Goods entering and leaving the system are equal, since $\lim_{\tau\to\infty} p^\tau = 1$. The amount of Money buyers start a Market with hence stabilizes to

$$M_b = m_b + \max(0, R_b - M_b).$$

This leads to the expected frustration being twice as high in the Sequence with the free market than in one with our distribution mechanism.

Theorem 4.8. *Consider a Sequence where traders follow the Greedy strategy. Then the expected frustration in the Market with the distribution mechanism is at most 1/2 of the free market's $\mathbb{E}_f$ as $\mathcal{T} \to \infty$.*

5 CONCLUSION

We present a novel multi-round trading environment, which combines free market and centralized distribution. The goal is that the equilibrium of the free market is moved towards the centralized solution, which is desirable in times of need. The system assigns the buyers a new commodity called *rights*, representing the amount of goods they are entitled to according to the centralized solution. The system trades both rights and goods. To evaluate the effectiveness of the redistribution, we use the concept of "frustration", which measures the gap between what a buyer obtained and what he was entitled to. We study two different implementations of the single-round market and show that in both cases, the frustration of each buyer in the multi-round system is upper bounded by 1/2.

Future work. We believe that our work has two major limitations. Firstly, we have focused on the most severe crisis scenarios and assumed a steady, albeit small, resupply of goods and money over multiple trading rounds. We would like to further explore the impact of more complicated system dynamics, such as the bullwhip effect that occurs when there is a sudden surge in demand at the onset of a crisis. Secondly, our results are not specific to any particular right distribution mechanism. Examining specific mechanisms may alter the system dynamics and potentially improve the bounds on the expected frustration. Additionally, it may be possible to design tailored rights distributing mechanisms respecting, e.g., certain fairness rules, using similar approaches as those employed for voting mechanisms, as seen in [7].

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A REMARKS

A.1 On the trading environment

To the Use of Money. There needs to be a protocol for maintaining Money obtained by selling Right. It is an ethical requirement that such Money remains in the system and is used for buying Good or Right during the current Crisis. We also require that the obtained Money is used in *a future Market, not the current* one, in order to give an advantage to the active buyers, who are along with the sellers the "kings" of the Crisis. The advantage, compared to the centralized solution, consists of obtaining Good earlier than the passive buyers. Another reason to have Money as a commodity is that the utility for Money differs among the participants and we will use this fact for studying our multi-round trading environment[5].

To the Distribution Mechanism. Our distribution mechanisms are designed with the consideration of various fairness rules, which govern the subsequent reallocation of goods desired by the central authority. However, we do not restrict ourselves solely to fairness mechanisms, as we believe in keeping the limitations to a reasonable minimum. During a crisis, which can take various forms, the authorities may adopt different rights distributions by optimizing different functions based on the crisis's specific features. From an ethical standpoint, an approach maximizing social welfare is preferred for a critical distribution, where we consider other parameters of the current crisis, not only the need itself. In addition, for similar actors with similar needs, a uniform approach is preferred. From a welfare perspective, an egalitarian distribution of welfare does not necessarily mean distributing scarce resources equally, as equality of welfare differs from equality of resources and is morally valuable as a goal. Therefore, our approach aligns with both egalitarian ethical perspectives, emphasizing the equality of welfare in society, and utilitarian ethical perspectives, which prioritizes maximizing the benefits generated by scarce resources and overall welfare for society members.

A.2 On the system with myopic traders

Feasibility of the complexity assumption of Theorem 3.3. All utility functions are known and monotone, $u_b(0, x)$ is linear and $u_b(x, 0)$ is concave. Let X be the current basket of buyer b. In the algorithm, buyer b does not need to know S^b, it only needs to

1. Find maximum k such that $u_b(\mathcal{G}(X)+k, 0)-u_b(\mathcal{G}(X), 0) > u_b(0, ck)$, thus needs to solve: given a constant K, find max k such that $u_b(\mathcal{G}(X) + k, 0) - u_b(\mathcal{G}(X), 0) > Kk$; *we assume that this can be done in a constant time by each buyer.*
2. Buy at most k items of Couple from buyers, satisfying the invariant of OUTBID.

A.3 On the system with non-myopic traders

To Intentional Storing. The Greedy strategy is for sellers to offer at most the amount of Good they receive at the beginning of each Market, as if the Good is perishable. At first, this seems to force sellers to store Good, but that is not the case as if all traders follow the Greedy strategy all Good is sold. However, it prevents certain deviations of buyers from Greedy to be beneficial. See Theorem 4.6 for more details.

To the Correctness of Definition 4.4. It may not be immediately clear how a seller can derive p^τ from Eq. (2), as it requires knowledge of R^τ_B. However, if all sellers adopt the Greedy strategy, the offered volume is $\sum_{s\in S} g_s = 1$, and the distribution mechanism is publicly available. In contrast, b does not know M^τ_{-b}, and must therefore estimate the price of the Right by averaging the selling price of the Good.

To the Earnings in the Following Market. In the following Market, the useless Money is allocated to buyers who were poor in the previous Market. Specifically, in the following Market a buyer will have

$$M_b^{\tau+1} = m_b + q^\tau R_b^\tau f_b^\tau = m_b + \frac{q^\tau}{p^\tau} \max(0, p^\tau R_b^\tau - M_b^\tau), \quad (6)$$

or, in other words, they earn a portion of the Money proportional to their frustration f_b^τ (see Definition 2.5) in this Market.

B PROOFS

B.1 Myopic traders

Theorem 3.3. *Let* $0 < \epsilon < 1$. *We assume that initial endowments of buyers are feasible for* ϵ *and the offered volume of Good is also feasible. We further assume that each buyer in every round does one algorithmic step. The following holds.*

1. *The time-complexity of the auction-based algorithm is at most* $|B|^2(1 + \log_{1+\epsilon} m)$*; hence, the auction-based algorithm is polynomial in the input size.*
2. *For each participant, its basket assigned by the algorithm is feasible and its price plus* 1 *is bigger than the total price of its initial endowment.*
3. *The terminal price of Right is equal to the terminal price of Good.*
4. *Relative to terminating prices, each buyer or seller gets a basket of utility at least* $(1 - 2\epsilon)$ *times the utility of its optimal feasible basket.*

We denote by m, r, g the total initial endowments of Money, Right and Good and recall that $r = g$. We prove the theorem in a sequence of lemmata.

Lemma B.1. *At each stage of the algorithm, the total amount of Cash among the buyers is at most* 2m. *We recall Cash is a dummy commodity introduced in subsection 3.1 which represents the flow of Money in the system.*

Proof. Lemma is true in the beginning by assumption (1) of Definition 3.2. The total amount of Cash is gradually decreasing during each iteration, since the system always buys for less than it sells. At the end of each iteration, the system gives back to the buyers the amount of Cash it earned during that iteration. ◻

Lemma B.2. *In the first iteration, all items of Good and Right are paired.*

[5] For a similar treatment of Money, see [6].

Proof. By assumptions (2) of Definition 3.2, all buyers prefer to buy at least the fair amount of items of Couple for the initial price; by assumption (1) there is enough cash in the initial surplus of each buyer to do it. □

Lemma B.3. *After the end of the first iteration: (1) a buyer owes to the system only cash for items of Money in its initial endowment and (2) total cash among participants is always at most m.*

Proof. The first part follows from Lemma B.2 since all items of Good and Right are sold and bought at the end of the first interaction. For the second part we note that among sellers, the total cash is pg since all items of Good were sold in the first iteration and among buyers, the total cash is at most $m - pg$ since the buyers paid for the items of Good and there is no cash left from the initial endowments of Right since all items of Right were sold and bought in the first iteration. □

Lemma B.4. *The number of rounds in an iteration is at most $2 + |B|$.*

Proof. We observe that in each fully completed round, either none of the buyers buys items of Couple and the trading ends, or none of the buyers buys items of Couple in the next round and the trading ends or at least one buyer acts for the last time in this iteration: if in the current round every buyer buys items of Couple only from itself then in the next round nobody buys since nobody got additional cash. Hence let a buyer b buys items of Couple from another buyer in the current round. It means that b gets no additional cash in this iteration since it has no items of Couple for c, otherwise it would have to buy these first by the rules of the outbid and the current round is the last active round for b. □

Lemma B.5. *The total number of iterations is at most $1 + \log_{1+\epsilon} m$.*

Proof. Each iteration raises the price of Couple by the factor of $(1 + \epsilon)$ and the max price per unit of Couple cannot be bigger than the total surplus. □

Lemma B.6. *Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket.*

Proof. (1) Buyers owe nothing to the system since after the end of the trading they keep only the items of Money they can buy with their remaining cash.

(2) After the end of the trading and buying items of Money, each participant is left with the amount of cash less than 1 by the second part of Lemma B.3.

(3) The basket of each seller is optimal since all items of Good were sold.

(4) The only reason why the basket of a buyer b is not optimal is: For some items of Couple, b paid $(1 + \epsilon)c$ where c is the terminal price of Couple. Let x (y respectively) denote the total number of items of Couple (Money respectively) in b's *optimal basket*. The optimal basket is market clearing and thus $y + cx = M_b$ and the utility of the *optimal basket* of b is $u_b(x, 0) + u_b(0, y)$. By assumption (3) of Definition 3.2, $y > cx$ since otherwise

$$u_b(x, 0) + u_b(0, y) \leq u_b(cx, 0) + u_b(0, y) < u_b(0, M_b).$$

In b's *terminal basket*, there are x items of Couple and at least $y - \epsilon cx - 1$ items of Money. First, let $\epsilon cx \geq 1$. The utility of b's *terminal basket* is thus, using the assumption on the linearity of the utility of Money, at least $u_b(x, 0) + u_b(0, (1 - 2\epsilon)y) = u_b(x, 0) + (1 - 2\epsilon)u_b(0, y)$.

Secondly, let $\epsilon cx < 1$. The utility of b's *terminal basket* is thus at least $u_b(x, 0) + u_b(0, (y - 2))$ and the lemma holds since we assume $1 > \epsilon > 2/M_b$. □

Now we are ready to prove Theorem 3.3.

Proof. (1) It follows from Lemmas B.4, B.5 that the time complexity of the auction-based algorithm behaves asymptotically as $|B|^2(1 + \log_{1+\epsilon} m)$. We note that the extra factor of $|B|^2$ comes from the assumption of the theorem that each act of a buyer in a round counts as one algorithmic step, see Remark A2 for feasibility.

(2) follows from (1) of Lemma B.3, (3) follows from the description of the algorithm and (4) is Lemma B.6. □

Theorem 3.5. *In all but possibly the first Market of a measurable Crisis where trading is implemented by the auction-based algorithm of section 3.1, each individual frustration is at most $1/2$.*

Proof. Let Market $\mathbb{M}^\tau, \tau \geq 1$ of the Sequence end and let us consider the next Market $\mathbb{M}^{\tau+1}$. By the assumptions of the theorem, the auction-based algorithm repeats the steps of Market $\mathbb{M}^\tau$. We say that a buyer is frustrated in a specific Market if their frustration after the Market is non-zero. After the final step of the auction for $\mathbb{M}^\tau$, the willingness to pay of non-frustrated (in $\mathbb{M}^\tau$) buyers is saturated. However, frustrated (in $\mathbb{M}^\tau$) buyers continue buying Couple since they acquired additional funds in $\mathbb{M}^\tau$. Let b be such a frustrated buyer. Let b sold, in Market $\mathbb{M}^\tau$, n_b items of the Right for the total price z_b. Hence, b is willing to buy an additional number of items of Couple for z_b items of Money. Let S_b be the set of n_b items of Couple containing the items of Right buyer b sold so far in $\mathbb{M}^{\tau+1}$. Buyer b buys the Couple of S_b at an increased price which in turn frees funds of active buyers who may buy back.

Lemma B.7. *Frustration of b can only go down, and non-frustrated (in $\mathbb{M}^\tau$) buyers stay with zero frustration.*

Proof. The assertion holds since each frustrated buyer b can buy at most $|S_b|$ additional items of Couple even if the price is not increased from the final price of $\mathbb{M}^\tau$. Hence, b buys Couple only from S_b. The non-frustrated buyers may buy back Couple from sets S_b but not beyond these sets since their willingness to pay is the same as in $\mathbb{M}^\tau$: if Couple of all sets S_b is bought back by non-frustrated buyers, then the distribution of Couple is the same as at the end of $\mathbb{M}^\tau$ but the price is higher and thus no non-frustrated buyer is willing to buy more Couple. □

Hence, we only need to rule out the case that the final frustration of each frustrated buyer b is strictly bigger than

1/2. Let $0 < n'_b < n_b$ be such that $n_b - n'_b = R_b/2$ where we denote by R_b the number of assigned rights to b in $\mathbb{M}^\tau$ (and thus also in $\mathbb{M}^{\tau+1}$). Let us assume frustrated buyers slowly increase the price of Couple.

Lemma B.8. *When the price reaches double the price of Couple in $\mathbb{M}^\tau$, each frustrated buyer b (1) gains n'_b additional items of Couple and thus has frustration 1/2, (2) spends all z_b additional items of Money, (3) keeps $R_b z_b/n_b \geq z_b$ items of Money for sold Rights and (4) the non-frustrated buyers are not willing to buy back any of these items of Couple acquired by the frustrated buyers.*

Proof. When buying n'_b additional items of Couple of S, buyer b only needs to buy items of Good by which b spends all z_b additional items of Money it got from the previous Market $\mathbb{M}^\tau$:

- $2n'_b z_b/n_b$ items of Money for buying n'_b items of Good from S_b, and
- $(n_b - 2n'_b)z_b/n_b$ items of Money needed to increase the price of $R_b - n_b = 2(n_b - n'_b) - n_b$ items of Good b already has, since $n_b - n'_b = R_b/2$.

(3) follows since b sold $R_b/2$ items of Right and the price is doubled. Hence it remains to show (4). Non-frustrated (in $\mathbb{M}^\tau$) buyers from which b bought new items of Couple are not willing to buy back since: They obtain in total $2n'_b z_b/n_b$ items of Money for the sold n'_b items of Couple of S_b, but in order to increase the price further, $2(n_b - n'_b)z_b/n_b$ items of this obtained Money is needed to increase the price of the remaining $(n_b - n'_b)$ items of Couple in S_b. Clearly by the definition of n'_b and since $n_b \leq R_b$, $(n_b - n'_b) = R_b/2 \geq n'_b$. ◻

Summarizing: the final price of Couple in $\mathbb{M}^{\tau+1}$ is at most double of the price of Couple in $\mathbb{M}^\tau$ and a frustrated buyer has frustration at most 1/2 while the non-frustrated buyers remain non frustrated. ◻

B.2 Non-myopic traders

Theorem 4.6. *The Greedy strategies* (1) *and* (3) *form a coalition-proof equilibrium of a feasible Sequence of any length. The equilibrium can be computed efficiently.*

Proof. We prove the theorem as a sequence of lemmas. We begin with two useful lemmas, showing how the price changes with Money, and throughout the Sequence.

Lemma B.9. *Let the traders follow the Greedy strategy. Then p^τ, given as a solution of Eq. (2) is an increasing function of M^τ_b $\forall b \in B, \tau \in \{1, \dots \mathcal{T}\}$.*

Proof. Taking the derivative of both sides yields

$$\begin{aligned}\frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} &= \frac{\mathrm{d}}{\mathrm{d}M^\tau_{b'}} \sum_{b\in B} M^\tau_b - \max(0, p^\tau R_b - M^\tau_b)\\ &= 1 - \sum_{b\in B} \mathrm{sign}(p^\tau R_b - M^\tau_b)\left(\frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} R_b - 1\right)\\ &= 1 + \tilde{N} - \frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} \sum_{b\in B} \mathrm{sign}(p^\tau R_b - M^\tau_b) R_b,\\ \Rightarrow \frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} &= \frac{1+\tilde{N}}{1+\tilde{R}} \geq 1,\end{aligned}$$

where $\tilde{N} > 0$ is the number of poor buyers and $0 < \tilde{R} < \tilde{N}$ is the sum of their Right. ◻

Lemma B.10. *Let the traders follow the Greedy strategy. Then $p^{\tau-1} < 1 \Rightarrow p^\tau > p^{\tau-1}$, resp. $p^{\tau-1} > 1 \Rightarrow p^\tau < p^{\tau-1}$.*

Proof. Substituting Eq. (6) into (5) gives

$$\begin{aligned}\sum_{b\in B} m_b + \max(0, p^{\tau-1}R_b - M^{\tau-1}_b) - \max(0, p^\tau R_b - M^\tau_b) = p^\tau,\\ 1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M^{\tau-1}_b) - \max(0, p^\tau R_b - M^\tau_b) = p^\tau,\end{aligned} \tag{7}$$

If $p^{\tau-1} < 1$, then the useful Money also is

$$\sum_{b\in B} M^{\tau-1}_b - \max(0, p^{\tau-1}R_b - M^{\tau-1}_b) < 1,$$

or in other words

$$\sum_{b\in B} M^{\tau-1}_b < 1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M^{\tau-1}_b).$$

Going back to Eq. (7), we see that

$$\begin{aligned}1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M^{\tau-1}_b) = p^\tau + \sum_{b\in B} \max(0, p^\tau R_b - M^\tau_b),\\ \sum_{b\in B} M^{\tau-1}_b < p^\tau + \sum_{b\in B} \max(0, p^\tau R_b - M^\tau_b) = \sum_{b\in B} M^\tau_b.\end{aligned}$$

Using Lemma B.9 we get $p^\tau > p^{\tau-1}$. Similarly, we can show $p^{\tau-1} > 1 \Rightarrow p^\tau < p^{\tau-1}$. ◻

To demonstrate that the Greedy strategy is an equilibrium, we show that no deviation from it is beneficial for any trader. We consider changes in the offered volume of Good and Right first.

Lemma B.11. *Let all traders follow Greedy except for a seller who offers less Good in a Market and sells it in the following Market. Then his utility decreases as a consequence.*

Proof. Note that the scenario in which the seller deviates is denoted with a hat. Since the distribution mechanism is non-decreasing in the total offered volume, we get $\hat{R}^{\tau+1}_b \geq R^{\tau+1}_b$ for all buyers. For contradiction we also assume the deviation is beneficial for the seller, i.e. $\hat{p}^{\tau+1} \geq p^{\tau+1}$. Since at least one buyer didn't get the Money for selling Right in τ, we have

$$\sum_{b\in B} \max(0, \hat{p}^{\tau+1}\hat{R}^{\tau+1}_b - \hat{M}^{\tau+1}_b) > \sum_{b\in B} \max(0, p^{\tau+1}R^{\tau+1}_b - M^{\tau+1}_b). \tag{8}$$

Using Eq. (2) in both scenarios gives

$$\sum_{b\in B} M_b^{\tau+1} - \max(0, p^{\tau+1}R_b^{\tau+1} - M_b^{\tau+1}) = p^{\tau+1}, \tag{9}$$

$$p^{\tau}\mathcal{V} + \sum_{b\in B} M_b^{\tau+1} - \max(0, \hat{p}^{\tau+1}\hat{R}_b^{\tau+1} - \hat{M}_b^{\tau+1}) = \hat{p}^{\tau+1}(1+\mathcal{V}), \tag{10}$$

or in combination with Eq. (8)

$$p^{\tau}\mathcal{V} + p^{\tau+1} > \hat{p}^{\tau+1}(1+\mathcal{V}).$$

The price if s deviates is thus upper bounded by the weighted average of prices under Greedy. We can consider two cases.

(1) $p^{\tau} < 1$: Then by Lemma B.10, $p^{\tau+1} \geq p^{\tau} \Rightarrow p^{\tau+1} > \hat{p}^{\tau+1}$, which is a contradiction.
(2) $p^{\tau} > 1$: Again by Lemma B.10, $p^{\tau+1} \leq p^{\tau} \Rightarrow p^{\tau} > \hat{p}^{\tau+1}$. This means the seller could have sold Good in τ and increase his payoff.

□

Lemma B.12. *Let all traders follow Greedy except for a rich, resp. a poor buyer who buys, resp. sells less Right in a Market. Then his utility decreases as a consequence.*

Proof. If a buyer deviates in this way, it leaves sellers with more Good for the next Market, and the buyers with more Money. But when following Greedy, the sellers will offer the same volume of Good in the next, leading to the same distribution of Right as in the last Market. And, since the buyers now have more Money than by Lemma B.9, the price increases. Moreover, if b was poor in τ, then he would receive less Money from selling Right, limiting the amount of Good he can buy in the next Market

$$\hat{M}_b^{\tau+1} = m_b + \max(0, p^{\tau}(R_b - \mathcal{V}) - M_b^{\tau}) < M_b^{\tau+1}.$$

□

Deviating from Greedy by changing the price has a similar effect.

Lemma B.13. *Let all traders follow the Greedy strategy. Then no trader can increase his utility by changing the selling price of Good, resp. Right in a single Market.*

Proof. Let us start with $s \in S$ changing p_s^{τ}. There are two cases

(1) $p_s^{\tau} < p^{\tau}$: In this situation, s will get less Money in τ, which will stay in the system. However, in the following Market, the Money will be split proportionally to g_s, decreasing the utility of s.
(2) $p_s^{\tau} > p^{\tau}$: The acceptable price of Good is the average of the selling prices. This means s will not sell anything, decreasing his utility as was shown in Lemma B.11.

Similarly for $b \in B$

(1) $q_b^{\tau} < q^{\tau}$: When selling at a lower price, b will get less Money in the next Market, decreasing his utility. Furthermore, some rich buyers will be left with more Money, increasing the price in the following Market.
(2) $q_b^{\tau} > q^{\tau}$: The acceptable price of Right is $p^{\tau} = q^{\tau}$, so b will not sell any Right. This will again get him less Money and increase the price in the next Market.

□

Now, let us focus on coalition-proofness.

Lemma B.14. *There does not exist any coalition of traders that could increase their individual utilities by deviating from the Greedy strategy.*

Proof. Let us split the proof into three parts, depending on the composition of the coalition C.

(1) $C \subset B$: To increase the utility of buyers, they need to acquire more Good, meaning $T \setminus C$ will have less. Since $\mathcal{G}(B_s) = 0\ \forall s \in S$ when following Greedy, $B \setminus C$ need to acquire less. The only way C can accomplish that is if they don't buy Right from $B \setminus C$. But, similar to the proof of Lemma B.12, this would only lead to an increase in price, lowering the utility of $b \in C$.
(2) $C \subset S$: The sellers have utility for Money, and since following Greedy they obtain all Money the buyers have, $S \setminus C$ needs to get less Money. The proportion in which the sellers split the Money of buyers is given by p_s^{τ} and g_s, of which they can influence p_s^{τ}. But any price change will not decrease Money $S \setminus C$ get. Decreasing will leave some Money for the next Market, of which $S \setminus C$ gets a portion. Increasing the price will leave C with extra Good, decreasing their utility.
(3) $C \subset T$: Again, $S \cap C$ can only increase their utility if $S \setminus C$ get less Money. This can only be accomplished if $B \cap C$ don't buy from $S \setminus C$, so they accept a lower price, which is still larger than the selling price of $S \cap C$. But if the selling price is lower, and they are selling only to a subset of buyers, they cannot get more Money.

□

Finally, let us discuss the computational complexity of a Market utilizing the Greedy strategy.

Lemma B.15. *The Greedy strategy can be computed efficiently.*

Proof. The computation of the Greedy strategy given the price of Good can be done in constant time. What remains to show is that Eq. (2) can be solved efficiently. But that is the case since the price $p^{\tau} \in \mathbb{R}_0^+$, allowing us to split the interval into at most $|B| + 1$ intervals separated by points $\left\{\frac{M_b^{\tau}}{R_b^{\tau}}\right\}_{b\in B}$. In each interval, Eq. (2) can be partitioned into $|B| + 1$ price intervals, where in each the computation reduces to solving a simple linear equation

$$\sum_{b\in B} M_b^{\tau} - (p^{\tau}R_b^{\tau} - M_b^{\tau})\text{sign}(p^{\tau}R_b^{\tau} - M_b^{\tau}) = p^{\tau}\sum_{b\in B} R_b^{\tau}.$$

which can be solved efficiently. If its solution lies in the corresponding interval, it is a solution of Eq. (2). By Brouwer fixed-point theorem the solution is guaranteed to exist, since the price is upper bounded by the free market clearing price. The complexity is thus linear in the number of buyers. □

This concludes the proof of Theorem 4.6 □

Proposition 4.7. *Let all traders follow the Greedy strategy. Then the mapping of the current price to the next one is a non-expansive mapping on $\mathbb{R}$ with the L1 norm, resulting in the limiting price being one.*

Proof. In the first part of the proof, we eliminate simple cases when the price can be equal to one. In other cases, the mapping is a contraction, which we will show in the second part of this proof.

Let us begin with a statement about uniqueness.

Lemma B.16. *For any $M_b^\tau \geq m_b, R_b \in [0,1]$, Eq. (2) has a unique solution.*

Proof. For fixed Money and Right of buyers, the left-hand side is a concave, decreasing piece-wise linear function of the price. Since the right-hand side is linear, they cross at most one point. For $p^\tau = 0$, the left-hand side is $\sum_{b\in B} M_b^\tau \geq \sum_{b\in B} m_b = 1$, while the right is zero, so a solution exists. □

It may happen that the price is one in a Market. But if that happens, the price stays for the remainder of the Sequence.

Lemma B.17. *Let all traders follow the Greedy strategy. Then $p^\tau = 1 \Rightarrow p^{\tau+1} = 1$.*

Proof. If $p^\tau = 1$, then

$$\sum_{b\in B} M_b^\tau - \max(0, R_b - M_b^\tau) = 1,$$

and in the next Market

$$\sum_{b\in B} M_b^{\tau+1} - \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) = p^{\tau+1},$$

but

$$\begin{aligned}\sum_{b\in B} M_b^{\tau+1} &= \sum_{b\in B} m_b + \max(0, R_b - M_b^\tau)\\ &= 1 + \sum_{b\in B} \max(0, R_b - M_b^\tau) = \sum_{b\in B} M_b^\tau.\end{aligned}$$

However, this implies that $p^{\tau+1} = p^\tau$. For contradiction, let B' be a set of poor buyers at τ. The price can only be influenced by the Money poor buyers have, since R_b is fixed. There are two options

(1) $p^{\tau+1} > p^\tau$: Then the poor buyers have more Money $\sum_{b'\in B'} M_{b'}^{\tau+1} > \sum_{b'\in B'} M_{b'}^\tau$. But that is not possible without increasing the amount of Money all buyers have, since for the rich $M_b^{\tau+1} = m_b$.
(2) $p^{\tau+1} < p^\tau$: In this case, the amount of Money the poor buyers have decreases, meaning for some $b \in B \setminus B'$, $M_b^{\tau+1} > m_b$. But for rich buyers $M_b^{\tau+1} = m_b$.

□

This result shows that the mapping is non-expansive if at some τ, $p^\tau = 1$. In other cases, we show the mapping is a contraction, i.e.

$$1 > \frac{|p^{\tau+1} - 1|}{|p^\tau - 1|}.$$

The rest of the proof of Proposition 4.7 is dedicated to proving this. Combining Eq. (2) and (6) we get

$$p^\tau - 1 = \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau),$$

so the contraction condition is

$$\begin{aligned}1 &> \frac{|p^{\tau+1} - 1|}{|p^\tau - 1|}\\ &= \left| \frac{\sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau) - \max(0, p^{\tau+1} R_b - M_b^{\tau+1})}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau)} \right|\\ &= \left| \frac{\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) - \max(0, p^\tau R_b - M_b^\tau)}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau)} \right|\\ &= \left| 1 - \frac{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^{\tau+1} R_b - M_b^{\tau+1})}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau)} \right|.\end{aligned} \tag{11}$$

We split the remainder of the proof into two parts

(1) The first option to satisfy Eq. (11) is

$$\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) < \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}), \tag{12a}$$

$$\sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau) < \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}). \tag{12b}$$

Eq. (12b) gives

$$\begin{aligned}\sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau) &< \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}),\\ \sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau) &< \sum_{b\in B} M_b^\tau - m_b,\\ -p^\tau + \sum_{b\in B} M_b^\tau &< -1 + \sum_{b\in B} M_b^\tau,\\ 1 &< p^\tau.\end{aligned} \tag{13}$$

Then the Eq. (12a)

$$\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) < \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}),$$

$$\sum_{b\in B} \max(0, p^{\tau+1} R_b - m_b - \max(0, p^\tau R_b - M_b^\tau)) < \sum_{b\in B} M_b^\tau - m_b.$$

Let us show that the inequality holds for all buyers, not just their sum. If b is poor in τ, then the inequality becomes

$$\max(m_b - M_b^\tau, (p^{\tau+1} - p^\tau) R_b) < 0,$$

however $M_b^\tau > m_b$ for a poor buyer, and using Eq. (13) and Lemma B.10 yields $p^{\tau+1} < p^\tau$.
Otherwise $M_b^\tau = m_b$ and $p^\tau R_b - M_b^\tau \leq 0$. Using Lemma B.10 we get $p^\tau R_b - M_b^\tau > p^{\tau+1} R_b - m_b$, so

$$\max(0, p^{\tau+1} R_b - m_b) < 0,$$

or b is rich in the next Market as well.

(2) Similarly, the second option is

$$\sum_{b\in B} \max(0, p^{\tau+1}R_b - M_b^{\tau+1}) > \sum_{b\in B} \max(0, p^{\tau-1}R_b - M_b^{\tau-1}),$$

$$\sum_{b\in B} \max(0, p^{\tau}R_b - M_b^{\tau}) > \sum_{b\in B} \max(0, p^{\tau-1}R_b - M_b^{\tau-1}),$$

where the second condition similarly reduces to $1 > p^{\tau}$. Focusing on the first one gives

$$\sum_{b\in B} \max(0, p^{\tau+1}R_b - m_b - \max(0, p^{\tau}R_b - M_b^{\tau}) > \sum_{b\in B} M_b^{\tau} - m_b. \tag{14}$$

Again, for a poor buyer in τ we get

$$\max(m_b - M_b^{\tau}, (p^{\tau+1} - p^{\tau})R_b) > 0$$

which is again true. For a rich buyer we get $\max(0, p^{\tau+1}R_b - m_b)$, which can be zero. However, since for the poor buyers the inequality was strict, Eq. (14) is satisfied.

□

THEOREM 4.8. *Consider a Sequence where traders follow the Greedy strategy. Then the expected frustration in the Market with the distribution mechanism is at most 1/2 of the free market's* $\mathbb{E}_f$ *as* $\mathcal{T} \to \infty$.

PROOF. When investigating frustration, we can focus only on the poor buyers, for whom $R_b > M_b$. Asymptotically, Eq. (6) thus becomes

$$M_b = \frac{m_b + R_b}{2},$$

corresponding to the same amount of Good they can buy, since the price is equal to one. This means their frustration is

$$f_b = \frac{R_b - \frac{m_b + R_b}{2}}{R_b} = \frac{1}{2}\left(1 - \frac{m_b}{R_b}\right) \le \frac{1}{2}.$$

In contrast, since in the free market $M_b = m_b$, the frustration of poor buyers is twice as high, i.e.,

$$f_b = \frac{R_b - m_b}{R_b} = 1 - \frac{m_b}{R_b}.$$

For each poor buyer, the frustration is asymptotically half of what it would be in the free market. Since the frustration of the rich buyers is zero, the overall expected frustration is asymptotically half of what it would be in the free market, once the initial portion of the Sequence becomes insignificant. □

C MARKET MECHANISM FOR NON-MYOPIC TRADERS

In this section, we provide a formal description of the market mechanism used in Section 4. We split Definition 4.1 into two parts. First, we define a notion of a compatible buyer with a set of buyers and sellers as a buyer who is willing to trade with the others. Later, using this notion, we define the interaction between compatible traders.

DEFINITION (COMPATIBLE BUYER). *Let* (v_s^{τ}, p_s^{τ}) *and* $(w_b^{\tau}, q_b^{\tau}, \overline{v}_b^{\tau}, \overline{p}_b^{\tau}, \overline{w}_b^{\tau}, \overline{q}_b^{\tau})$ *be the bids of sellers and buyers respectively. A buyer* $b \in B$ *is said to be compatible with offers of* $S' \subset S$ *if* $\overline{p}_b^{\tau} \ge p_{s'}^{\tau}, R_b^{\tau} > 0$ *and* $v_{s'}^{\tau} > 0\ \forall s' \in S'$. *We denote the set of buyers compatible with offers of sellers in* S' *as* $C_1(S')$.

Similarly, a buyer b *is compatible with offers of sellers in* S' *and buyers in* $B' \subset B \setminus \{b\}$ *if* $\overline{p}_b^{\tau} \ge p_{s'}^{\tau}, \overline{q}_b^{\tau} \ge q_{b'}^{\tau}$ *and* $v_{s'}^{\tau}, w_{b'}^{\tau} > 0\ \forall s' \in S', b' \in B'$. *We denote the set of buyers compatible with offers of sellers in* S' *and buyers in* B' *as* $C_2(S', B')$.

DEFINITION (MARKET MECHANISM). *The market mechanism is a function* $\mu : \Pi \times \mathbb{R}_0^{+,2|T|+|B|} \to \mathbb{R}_0^{+,2|T|}$, *written as*

$$\mu(v_S^{\tau}, p_S^{\tau}, w_B^{\tau}, q_B^{\tau}, \overline{v}_B^{\tau}, \overline{p}_B^{\tau}, \overline{w}_B^{\tau}, \overline{q}_B^{\tau}, G_T^{\tau}, M_T^{\tau}, R_B^{\tau}) = B_T^{\tau},$$

where B_t^{τ} *is a basket containing the amount of Good and Money* t *gained during trading. The market mechanism we consider has two stages. In the first stage, the buyers use the Right they were assigned to buy as much Good as they desire. In the second stage, the buyers buy Good and Right in equal volume, until they buy their desired volume of either, or they have no Money left. In both stages, items offered at a lower price are traded first. When more traders offer Good or Right at the same price, they are treated as a single trader until one runs out of items for sale.*

The structure of the market mechanism is outlined in the Algorithm 1, using the notion of compatible buyer C_1 *and* C_2, *defined above. The overall Market game can be found in Algorithm 2, and the Sequence in Algorithm 3.*

D CANONICAL SOLUTIONS

In this section, we study a solution of the Sequence with perhaps the simplest class of distribution mechanism. Specifically, this distribution mechanism assigns all Right to a buyer with the n^{th} largest Claim[6]

$$\varphi_{b,n}\left(\sum_{s\in S} v_s^{\tau}, D\right) = \begin{cases} \sum_{s\in S} v_s^{\tau} & \text{if } D_b \text{ is the } n^{\text{th}} \text{ highest Claim,} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

We call it the *n-canonical distribution mechanism.* Let b_n be the buyer receiving all Right. In $\tau = 1$ we get

$$\sum_{b\in B} M_b^1 - \max(0, p^1 R_b - M_b^1) = p^1 \sum_{b\in B} R_b,$$

$$\sum_{b\in B} m_b - \max(0, p^1 R_b - m_b) = p^1,$$

$$1 - p^1 + m_{b_n} = p^1, \quad \Rightarrow \quad p^1 = \frac{1 + m_{b_n}}{2}. \tag{16}$$

Therefore, in the second Market, $M_{b_n}^2 = m_{b_n} + \frac{1+m_{b_n}}{2} - m_{b_n} = \frac{1+m_{b_n}}{2}$ and the price is

$$\sum_{b\in B} M_b^2 - \max(0, p^2 R_b - M_b^2) = p^2 \sum_{b\in B} R_b,$$

$$1 + \frac{1 + m_{b_n}}{2} - m_{b_n} - p^2 + m_{b_n} + \frac{1 + m_{b_n}}{2} - m_{b_n} = p^2$$

$$\Rightarrow p^2 = \frac{2 + 2m_{b_n}}{2} - m_{b_n} = 1.$$

This means b will have $M_b^3 = m_b + 1 - \frac{1+m_b}{2} = \frac{1+m_b}{2} = M_b^2$ and the cycle repeats.

[6] Ties are broken arbitrarily, but consistently.

Algorithm 1: Market mechanism

$\Delta G_t^\tau, \Delta M_t^\tau \leftarrow 0 \; \forall t \in T$
$g \leftarrow \mathrm{sort}(\mathrm{unique}(\{p_s | s \in S\}))$ `// Sorted list of unique prices of Good`
$r \leftarrow \mathrm{sort}(\mathrm{unique}(\{q_b | b \in B\}))$
`/* First stage when buyers use their Right */`
for p *in* g **do**
  $S' \leftarrow \{s | s \in S, p_s = p\}$ `// Set of sellers offering Good at a given price`
  while $|C_1(S')| > 0$ **do**
    $v^\tau \leftarrow \sum_{s \in S'} v_s^\tau$ `// Total offered volume at this price`
    $\overline{v}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')}(M_b^\tau / p, R_b^\tau - w_b^\tau, \overline{v}_b^\tau)$ `// Total desired and affordable volume`
    $V^\tau \leftarrow \min(v^\tau, \overline{v}^\tau)$
    $\Delta G_s^\tau \leftarrow \Delta G_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
    $v_s^\tau \leftarrow v_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
    $\Delta G_b^\tau \leftarrow \Delta G_b^\tau + V^\tau / |C_1(S')|, \; \forall s \in S'$
    $R_b^\tau \leftarrow R_b^\tau - V^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
    $M_b^\tau \leftarrow M_b^\tau - pV^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
    $\overline{v}_b^\tau \leftarrow \overline{v}_b^\tau - V^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
`/* Second stage when buyers buy Right and Good in equal quantity */`
for p *in* l **do**
  for q *in* r **do**
    $S' \leftarrow \{s | s \in S, p_s = p\}$
    $B' \leftarrow \{b | b \in B, q_b = q\}$
    while $|C_2(S', B')| > 0$ **do**
      $v^\tau \leftarrow \sum_{s \in S'} v_s^\tau$ `// Total offered volume of Good at this price`
      $w^\tau \leftarrow \sum_{b \in B'} w_b^\tau$ `// Total offered volume of Right at this price`
      $\overline{v}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')}(M_b^\tau / (p+q), \overline{v}_b^\tau)$
      $\overline{w}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')}(M_b^\tau / (p+q), \overline{w}_b^\tau)$
      $V^\tau \leftarrow \min(v^\tau, \overline{v}^\tau, w^\tau, \overline{w}^\tau)$ `// Total volume to be traded`
      $\Delta G_s^\tau \leftarrow \Delta G_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
      $v_s^\tau \leftarrow v_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
      $w_b^\tau \leftarrow w_b^\tau - V^\tau / |S'|, \; \forall b \in B'$
      $\Delta G_b^\tau \leftarrow \Delta G_b^\tau + V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $M_b^\tau \leftarrow M_b^\tau - (p+q)V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $\overline{v}_b^\tau \leftarrow \overline{v}_b^\tau - V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $\overline{w}_b^\tau \leftarrow \overline{w}_b^\tau - V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
return: $(G_T^\tau + \Delta G_T^\tau, M_T^\tau + \Delta M_T^\tau)$

Algorithm 2: Market

input: $G_t^\tau, M_t^\tau, \; \forall t \in T$
$(v_s^\tau, p_s^\tau) \leftarrow \pi_s(G_B^\tau, M_B^\tau, G_S^\tau), \; \forall s \in S$
$R_B^\tau \leftarrow \phi(\sum_{s \in S} v_s^\tau, D_B)$
$(w_b^\tau, p_b^\tau) \leftarrow \hat{\pi}_b(v_S^\tau, p_S^\tau, G_B^\tau, M_B^\tau, R_B^\tau), \; \forall b \in B$
$(w_b^\tau, p_b^\tau, \overline{v}_b^\tau, \overline{p}_b^\tau, \overline{w}_b^\tau, \overline{q}_b^\tau) \leftarrow$
  $\pi_b(v_S^\tau, p_S^\tau, G_B^\tau, M_B^\tau, R_B^\tau, w_{-b}^\tau, q_{-b}^\tau), \; \forall b \in B$
$B_T^\tau \leftarrow \mu(v_S^\tau, p_S^\tau, w_B^\tau, q_B^\tau, \overline{v}_B^\tau, \overline{p}_B^\tau, \overline{w}_B^\tau, \overline{q}_B^\tau, M_B^\tau, R_B^\tau)$
$u_T^\tau \leftarrow u(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau))$
return: $(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau), u_T^\tau)$

Any distribution mechanism ϕ can be constructed as a sum weighted sum of canonical mechanisms

$$\phi(V, D) = \sum_{n=1}^{|B|} \alpha_n \varphi_{b,n}(V, D). \tag{17}$$

Algorithm 3: Sequence

$G_t^1, M_t^1, u_t \leftarrow 0, \; \forall t \in T$
for $\tau \in \{1, \dots \mathcal{T}\}$ **do**
  $\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau), u_T^\tau \leftarrow \mathbb{M}(G_T^\tau, M_T^\tau)$
  $u_t \leftarrow u_t + u_t^\tau, \; \forall t \in T$
  $G_t^{\tau+1}, M_t^{\tau+1} \leftarrow \rho(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau)), \; \forall t \in T$
return: u_T

Since Eq. (2) is quasi-linear in this composition of Right

$$\sum_{b \in B} M_b^\tau - \max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - M_b^\tau) = p^\tau \sum_{b \in B} \alpha R_b^\tau + \beta \overline{R}_b^\tau,$$

$$\sum_{b \in B} (\alpha + \beta) M_b^\tau - \max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - (\alpha + \beta) M_b^\tau) = p^\tau \sum_{b \in B} \alpha R_b^\tau + \beta \overline{R}_b^\tau.$$

We want to split the maximum into two maxima, but

$$\max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - (\alpha+\beta)M_b^\tau) \\ \leq \alpha \max(0, p^\tau R_b^\tau - M_b^\tau) + \beta \max(0, p^\tau \overline{R}_b^\tau - M_b^\tau),$$

so computing each price separately gives a lower bound on the price in the composed system, which can be obtained by solving

$$\alpha \sum_{b\in B} M_b^\tau - \max(0, p^\tau R_b - M_b^\tau) + \beta \sum_{b\in B} M_b^\tau - \max(0, p^\tau \overline{R}_b - M_b^\tau) \\ = p^\tau \alpha \sum_{b\in B} R_b + p^\tau \beta \sum_{b\in B} \overline{R}_b$$

This approach can be further generalized to an arbitrary decomposition of the distribution mechanism, leading to a lower bound

$$\sum_n \alpha_n \sum_{b\in B} M_b^\tau - \max(0, p^\tau R_{b,n} - M_b^\tau) = \sum_n \alpha_n p^\tau.$$

This construction is especially useful, since we can solve the n problems separately. This can be done for the canonical distribution mechanism, leading to a lower bound on price in a Sequence with an arbitrary distribution mechanism.

Proposition (lower bound on price). *Let all traders follow the Greedy strategy. Then the price p^τ in Market $\mathbb{M}^\tau$ of a Sequence with distribution mechanism ϕ satisfies*

$$p^\tau \geq \frac{\sum_{b\in B}(\alpha_b + 1)M_b^\tau}{2}, \qquad \textit{where} \qquad \alpha_b = \phi_b(V, D).$$

Proof. Let M_b^τ be the amount of Money buyers have at the start of Market τ. We can write the distribution ϕ as

$$\phi(V, D) = \sum_{n=1}^{|B|} \alpha_n \varphi_{b,n}(V, D).$$

where $\varphi_{b,n}(V, D)$ is the n-canonical distribution mechanism defined in Eq. (15), and $\alpha_n \in [0, 1]$.

In the system with the canonical distribution mechanism we can obtain an explicit solution in a similar way to Eq. (16)

$$\sum_{b\in B} M_b^\tau - \max(0, p^\tau R_{b_n} - M_b^\tau) = p^\tau,$$

$$M_{b_n}^\tau - p^\tau + \sum_{b\in B} M_b^\tau = p^\tau,$$

$$\Rightarrow \quad p^\tau = \frac{M_{b_n}^\tau + \sum_{b\in B} M_b^\tau}{2}.$$

Weighing the prices according to the decomposition of the distribution mechanism ϕ, we obtain the statement of the proposition. □