Social Distancing by Autonomous, Possibly Territorial, Agents on Networks

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ABSTRACT

During epidemics, the population is asked to Socially Distance, with pairs of individuals keeping two meters apart. We model this as a new optimization problem by considering a team of agents placed on the nodes of a network. Their common aim is to achieve pairwise graph distances of at least D, a state we call *socially distanced.* (If $D = 1$, they want to be at distinct nodes; if $D = 2$, they want to be non-adjacent.) We allow only a simple type of motion called a Lazy Random Walk: with probability p (called the *laziness* parameter), they remain at their current node next period; with complementary probability $1 - p$, they move to a random adjacent node. The team seeks the common value of p, which achieves social distance in the least expected time, which is the absorption time of a Markov chain. We also consider a game where competitive agents seek to be alone at their own node before others are alone. Finally, we see how dispersion can be speeded up when the agents are territorial.

KEYWORDS

Autonomous agents, Epidemic, Random walk, Dispersion

1 INTRODUCTION

To combat epidemics, three actions are recommended to the public: mask wearing, hand washing, and social distancing. This paper models the last of these in an abstract model of autonomous mobile agents on a network. Social distancing can be considered a group goal (common-interest game) or an individual goal (antagonistic, non-cooperative game). We consider both goals in a dynamic model where agents (players) walk on a network (graph). A group of m players, or mobile agents, is placed in some way on the nodes of a network Q. Each agent adopts a Lazy Random Walk (LRW), which stays at his current node with a probability p (called laziness) and moves to a random adjacent node with complementary probability $1-p$ (called *speed*). In the common interest game, we seek a common value of p which minimizes the time for all pairs of players to be at least D nodes apart (socially distanced). Once p is adopted by all, the positions of the agents on the network (called states) follow a Markov chain, with distanced states as absorbing. Standard elementary results on absorption times for Markov chains are used to optimize p to find the value of p, which is adopted by all agents to minimize the absorption time. This work can be seen as an extension to networks of the *spatial dispersion* problem introduced by Alpern and Reyniers (2002), where agents could move freely between *any* two locations. Grenager et al (2002) extended that work to computer science areas, and Blume and Franco (2007) to economics. See also Simanjuntak (2014).

It is clear that ours is an extremely abstract approach to the problem of social distancing. For a very recent practical analysis of the impact of social distancing on deaths from Covid-19, including a monetary equivalent, see Greenstone and Nigam (2020). For mental health implications of social distancing, see Venkatesh and Edirappuli (2020), and for economic analysis, see Farboodi et al (2021).

This paper is organized as follows. Section 2 describes our dynamic model of agents moving on a network according to a common lazy random walk and derives the associated Markov chain. A formula for the time to absorption (desired state) is derived. Section 3 considers two agents attempting to social distance on the cycle graph. Section 4 considers three agents on the three-node cycle graph C_3. Section 5 considers a *game* where n competing players start together at one end of a line graph, each choosing their own laziness in a lazy random walk. In the first time period, where some agents are alone at their node, these agents split a unit prize. When $n = 2$ *any* pair (p, p) is an equilibrium, but when $n = 3$, there is no symmetric equilibrium. Some larger problems of social distancing are considered in section 6 using simulation. Section 7, taken from Zeng (2023), analyzes the effect of territoriality of the agents on dispersion times. Section 8 concludes. Aside from Section 7, this paper is taken directly from Alpern and Zeng (2022). Some proofs omitted here can be found there.

2 STATES AND LAZY RANDOM WALKS

The m agents in our model move on a connected network Q with n nodes, $n \geq m$, labelled 1 to n. We take a graph theoretic assumption where all arc lengths are 1, so we will, from now on, call Q a graph.

2.1 States of the system

There are several ways to denote the *state* of the system. A general way is to write square brackets $[j_1, j_2, \ldots, j_n]$, where j_i is the number of agents at node i, with $\sum_{i=1}^{m} j_i = m$. We call the number j_i the *population* of node i. Attached to every state is a number d denoting the minimum distance between two agents, where we use the graph distance between nodes (the number or arcs in the shortest path). For example, if we number the nodes of the line graph L_6 consecutively, and the state is $[1, 0, 0, 1, 0, 1]$, then $d = 2$. If a state has distance d, it is called *socially distanced* if $d \geq D$, where D is a parameter of the problem. For example, the state $[1, 0, 0, 1, 0, 1]$ is socially distanced for $D = 1$ and $D = 2$ but not for $D = 3$. For the social distancing problem, the states with $d \geq D$ are considered the absorbing states.

2.2 Lazy Random Walks

The unifying idea of this paper is the use of agent motions of the following type.

Definition 1. *A Lazy Random Walk (LRW) for an agent on the graph Q, with laziness parameter p (and speed $q = 1 - p$) is as follows. With probability p, stays at your current node. With probability $q = 1 - p$, go equiprobably to any of the δ adjacent nodes, where δ is the degree of your current node. If $p = 0$, this is called simply a Random Walk. If the graph has constant degree Δ, then an LRW with $p = 1/(\Delta + 1)$, the process is called a Loop-Random Walk. That is because it would be a Random Walk if loops were added to every node. That is, all adjacent nodes are chosen equiprobably, including the current node.*

If all the agents in the model follow independent LRWs with the same value of p,, i.e. our main assumption, then a Markov chain is thereby defined on the state space $\mathcal{S}$. We only consider triples m (number of agents), D desired social distancing and Q (the connected graph), where it is possible to have distanced states. For example, the triple $m = 3$, $D = 2$ and $Q = C_5$ (cycle graph with 5 nodes) have no distanced states. In general, we assume that the $D-$Independence number (maximum number of D distanced nodes) is at least m. If $D = 1$, this is called simply the independence number. If $n = m$ and $D = 1$, we call this the spatial dispersion problem of Alpern and Reyniers (2002), an important special case of social distancing.

Given Q (with n nodes), m and D, there is a Markov chain on the state space $\mathcal{S}$ with a non-empty set of absorbing states $\mathcal{A}$. Suppose we number the non-absorbing states as $1, 2, \ldots, N$, and let B denote the $N \times N$ matrix where $b_{i,j}$ is the transition probability from state i to state j. Let t be the vector (t_i) denote the expected time (number of transition steps) to reach an absorbing state from state i. The t_i then satisfy the simultaneous equations

$$\begin{aligned} t_1 &= 1 + b_{11}t_1 + \cdots + b_{1j}t_j + \cdots + b_{1n}t_n \\ &\vdots \\ t_i &= 1 + b_{i1}t_1 + \cdots + b_{ij}t_j + \cdots + b_{in}t_n \\ &\vdots \\ t_n &= 1 + b_{n1}t_1 + \cdots + b_{nj}t_j + \cdots + b_{nn}t_n \end{aligned} \tag{1}$$

We can write this in matrix terms, where J_n the 1 by n matrix of 1s and I_N is the $N \times N$ identity matrix, as

$$\begin{aligned} t &= J_n + Bt, \text{ or} \\ (I_n - B)\, t &= J_n, \text{ with solution} \\ t &= (I_N - B)^{-1} J_n. \end{aligned}$$

So the solution for the absorption time vector t is given by

$$t = F J_n, \text{ where} \tag{2}$$

$F = (I_N - B)^{-1}$ is known as the *fundamental matrix*.

A useful variation is to allow agents to see the number k of agents at their node, the *population* of the node. In this case, it may allow a laziness p_k that depends on this k. For example, if I find myself at a node with three other agents, I stay there with probability p_4, which is a number that is part of the overall strategy. But generally, and unless stated, we assume there is only one value of p regardless of the population of the node.

3 SOCIAL DISTANCING ON $C_n, n \geq 4$ WITH $D = 2$

We begin with a simple example where two agents who start in adjacent nodes try to achieve distance $D = 2$ on a cycle graph C_n with $n \geq 4$ nodes.Cases $n \geq 5$ and $n = 4$ have different solutions. We take advantage of the symmetry of the cycle graph to use a reduced state space determined by the distance j between the agents. State j covers all configurations where this distance is $j - 1$, so that we have the usual row and column numbers for our matrices. The three states $j = 1, 2, 3$ are depicted in Figure 1 for both C_5 (top) and C_4 bottom. For both cases of n, there are (up to symmetry) two non-absorbing states (1 and 2) and a single absorbing state 3.

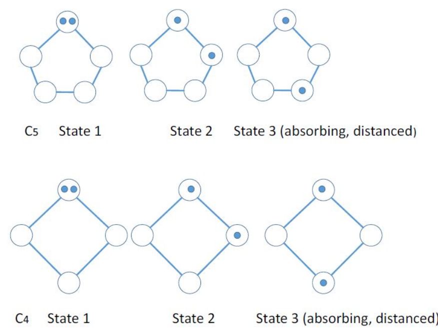

Figure 1: States for C_5 and C_4, $d = 2$.

To see the difference between $n \geq 5$ and $n = 4$, consider the (expected) absorption time T from state 2 when adopting a random walk (an LRW with $p = 0$). In C_n, $n \geq 5$, when both agents move from state 2, if they go in the same direction (probability 1/2) or towards each other (probability 1/4), they stay in state 2. If they go in opposite directions (probability 1/4), they reach the absorbing state 3. So T satisfies the equation

$$T = (3/4)\,(1 + T) + (1/4)\,(1) \implies T = 4. \tag{3}$$

However, in the graph C_4, if they start in state 2, they stay forever in state 2, so $T = \infty$. In the following two subsections on $n \geq 5$ and $n = 4$, we consider Population Dependent Lazy Random Walks, using the notation $p_1 = p$ (used when alone at a node - in state 2) and $p_2 = r$. We set $q = 1 - p$ and $s = 1 - r$ for the complementary probabilities. We solve this problem and then the simpler LRW problem by setting $p_1 = p_2$ ($p = r$).

3.1 The case of $C_n, n \geq 5$

The absorption times from states $i = 1, 2$ are

$$\begin{pmatrix} t_1\,(p, r) \\ t_2\,(p, r) \end{pmatrix} = F \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} \frac{14p+2}{(1-r)} + 16\frac{r}{(1-p)} \\ \frac{12r+4}{(1-p)} + 8\frac{p}{(1-r)} \end{pmatrix} \tag{4}$$

Since we are starting in state 2 we minimize $t_2\,(r, p)$ at

$$\begin{aligned} \bar{p}_2 &\equiv \bar{r} = 0,\ \bar{p}_1 = \bar{p} = \left(\sqrt{33} - 5\right)/2 \simeq 0.372\,28, \\ \bar{t}_2 &\equiv t_2\,(\bar{p}_2, \bar{r}_2) = \left(\sqrt{33} + 15\right)/8 \simeq 2.593\,1 \end{aligned}$$

3.2 The case of C_4

On the graph C_4, the transition matrix changes in the transition probability from state 2 to state 2 because if the agents move away from each other, the state remains state 2. The transitions among the non-absorbing states are now

$$B = \begin{pmatrix} r^2 + s^2/2 & 2rs \\ pq & p^2 + q^2 \end{pmatrix} \tag{5}$$

A similar analysis to that for $n \geq 5$ now shows that starting from either state 1 or state 2, the optimal strategies are $p_2 = r = 0$ and $p_1 = p = 1/2$. Assuming this, we have $\bar{t}_1 = 2$ and $\bar{t}_2 = 3$. This is counter-intuitive in that it is quicker to socially distance starting with both agents at the same node than starting with them at adjacent nodes. If we seek the optimal LRW, the solution depends on where we start. If we start at state 2 (two at a node), then it turns out that the random walk ($p = 0$) is optimal, with (as shown above) an absorption time of 4. We already know that a random walk starting at state 2 will never achieve social distancing, as in this case, state 2 will never be left. In this case the optimal p is $(1/10)\left(-1 + (49 - 20\sqrt{6})^{1/3} + (49 + 20\sqrt{6})^{1/3}\right) = 0.382\,72$. The absorption time for this LRW is approximately 4.45.

4 THREE AGENTS ON THE CYCLE GRAPH C_3

We consider how three agents placed on the nodes of C_3 can achieve social distancing with $D = 1$. This means that all pairwise distances must be at least 1, that is, the agents must occupy distinct nodes. This is also called the dispersion problem (one agent at each node). It turns out, surprisingly, that the initial placement of the agents does not affect the optimal strategy, which is the loop-random walk. The proof of the following result is in Alpern and Zeng (2022).

Proposition 2. *If three agents are placed in any way on the nodes of C_3, then the expected time to the social distanced state $j = 3$ (one on each node) is uniquely minimized when the agents adopt the loop-random walk ($p = 1/3$).*

5 NO EQUILIBRIUM IN FIRST-TO-DISPERSE GAME

In this section we consider the game $G_1(n)$, where n players start together at the end location 1 on the line graph L_n with nodes $1, 2, \ldots, n$. When some players first achieve "ownership" of a node (are alone at their node), these players equally split a prize of 1. Each player i has a single strategic variable, her laziness probability p_i. We seek symmetric equilibria (with all p_i the same) for the cases $n = 2, 3$.

We can consider this game as a selfish form of the social distancing problem with $D = 1$ and $m = n$ (so it is also a dispersion problem) on the line graph L_n. In a version of this problem with what we call *territoriality*, a player who is alone at her node becomes the *owner* of it. This means she stays there forever, and anyone else who lands there immediately moves away randomly in the next period. So, the game considered here can be thought of as the beginning of a dispersal problem with territoriality.

When $n = 2$, this is an almost trivial case. For any $p \in (0, 1)$, the game eventually ends with probability one (as soon as one player moves and one stays in the same period), with a payoff of 1/2, since both players will achieve ownership at the same time. So *any* pair (p, p) is a symmetric equilibrium.

The case $n = 3$ is more complicated, and the proof is in Alpern and Zeng (2022). By symmetry, it is clear that when all players adopt stay probability p,, they all have an expected payoff of 1/3. We will show that when any two players adopt the same p, the remaining player can get more than 1/3 by a suitable strategy, and hence there is no symmetric equilibrium. The algebra involved in the proof is greatly simplified if we consider the "modified payoff" $M(q, p)$ to the single player (call her Player 1) adopting q when the other two adopt p. It is modified from the actual payoff by not giving her the prize of 1/3 when there is a tie. So it will be enough to show that Player 1 can always find a q (for any p adopted by the others) with $M(q, p) \geq 1/3$ when a tie is possible, and consequently, her actual payoff will strictly exceed 1/3. So no triple (p, p, p) can constitute an equilibrium.

Theorem 3. *There is no symmetric Nash equilibrium for the game $G_1(3)$.*

The proof for Theorem 3 can be found in Alpern and Zeng (2022). We refer the interested reader to that paper for the complete proof.

6 SIMULATION OF SOCIAL DISTANCING ON THE LINE AND GRID GRAPHS

For larger problems with respect to m and n,, we determine the expected time to reach social distancing with $D = 2$ by simple Monte Carlo simulation methods. We place the m agents in some specified initial locations on the network. Then we have them move independently according to LRWs with the same p value. After each step, we find the minimum pairwise distance d between agents in the current state. If $d \geq D$ (= 2 for the examples here), we stop and record the time T. We carry out 5,000 trials and record the mean. Contrary to our earlier results, we find for the line and the two-dimensional grid that it is optimal for the agents to follow (independent) random walks, $p = 0$. When n is very small, it takes a little longer to reach social distancing. While we focus on the left corner start and the center start here, it is worth noting that the situation where the agents are initially placed randomly on the nodes has been analysed for some graphs in Alpern and Zeng (2022).

6.1 The two dimensional $k \times k$ grid GR_k

In practice, social distancing is often to be achieved by individuals in a planar region. A good network model for this is the two-dimensional grid graph GR_k with $n = k^2$ nodes in the set $\{(i, j\} : 1 \leq i, j \leq k\}$, as shown in Figure 2.

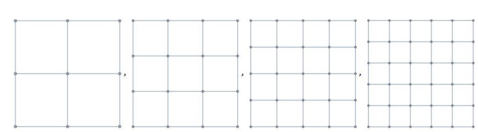

Figure 2: Two dimensional grids GR_k, $k = 3, \ldots, 6$.

A natural starting state is the one with all agents at a corner node (say $(1, 1)$) or at the center (both coordinates $\lfloor k/2 \rfloor$. Figure

14 illustrates these times for values of p spaced at a distance of 0.2. Note that for all the four values of k, the mean times to reach distance $d = 2$ are increasing in p. This means that the random walk, $p = 0$, is the best. In terms of grid size k, It takes a bit longer for the 3×3 grid because reflections from the boundary are more common. For larger values of k, the times do not appear to depend much on k.

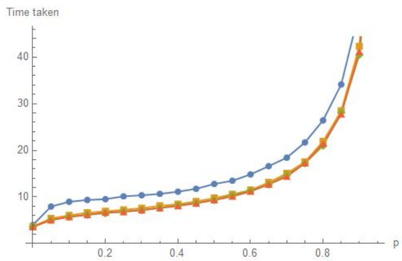

Figure 3: Time to $d = 2$ on GR_k, $k = 3$ to 6, from a corner start.

If the starting state consists of all agents at the center of the grid, then we have a similar result, as seen in Figure 4.

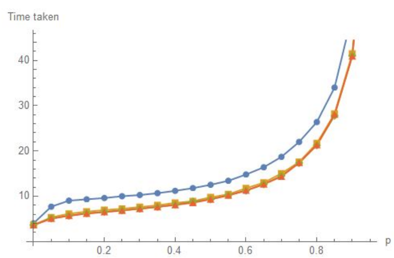

Figure 4: Time to $d = 2$ on GR_k, $k = 3$ to 6, from a center start.

6.2 The line graph L_n

The graph L_n has n nodes arranged in a line and numbered from the left as 1 to n. Like the grid graph, a natural starting state is either all at an end (say node 1) or all at the center. We find that the common value of p should be 0, that is, the agents should adopt independent random walks. Figure 5 shows this for a left start, and Figure 6 shows this for a center start, at $\lfloor n/2 \rfloor$.

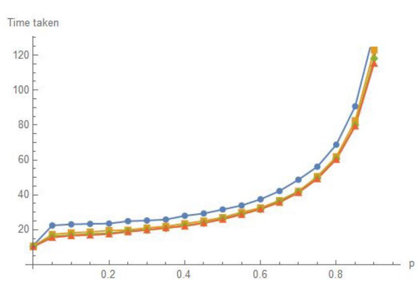

Figure 5: Time to $d = 2$ for left node start on L_n.

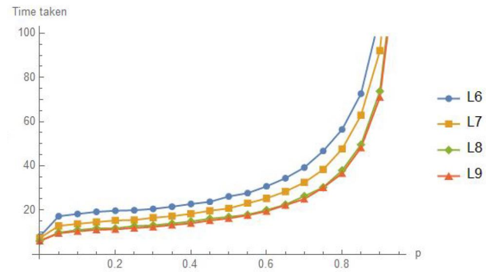

Figure 6: Time to $d = 2$ for a central start.

7 DISPERSION PROBLEM WITH TERRITORIAL BEHAVIOUR

Territorial behaviour happens when an animal consistently defends its territory from incursions by others. This behaviour is an interesting topic in habitat selection and has been studied by Fretwell (1970), Danchin, Giraldeau, and Cézilly (2008). They found that animals would avoid coming to an area with a high density of previous animals. As an extension, we now study the effect of territorial behaviour at the social distancing problem with $D = 1$ and $m = n$ (also called dispersion problem) on the networks by simple Monte Carlo simulation methods. We define agents' strong territorial behaviour as: if an agent is alone on a node for the first time, this agent then *owns* the node and stays there forever. Any other agent that comes to a node which is owned will be pushed away and will move to the adjacent nodes randomly in the next period.

Agents are placed together initially at the leftmost location of a line graph L_n. Then we have them move independently according to LRWs with the same p value if they are not alone. Once an agent is alone on a node, he will be marked as the owner of this node. Then he will stay with probability one forever, and all the other agents who come to nodes with owners will move randomly to an adjacent node with probability one in the next step. After each step, we find the minimum pairwise distance d between agents in the current state. If $d \geq D$ (= 1 for the examples here), we stop and record the time T. We carry out 5,000 trials and record the mean.

7.1 The case of L_3 and L_4

On the graph L_3 and L_4, the strong territorial behaviour significantly decreases the expected time for all agents to reach the absorbing state (also called the dispersion time). For L_3, as seen in Figure 7, without strong territoriality, the minimum dispersion time is 6.14 with optimal $p = 0.22$ and with territoriality, the minimum dispersion time is 4.55 with optimal $p = 0.24$. The optimal p are similar on L_3, but the strong territoriality decreases the dispersion time by 26%. For L_4, the optimal p are now quite different. Without strong territoriality, the minimum dispersion time is 16.74 with optimal $p = 0.31$; with territoriality, the minimum dispersion time is 9.45 with optimal $p = 0.10$. The optimal p is much smaller when agents have strong territoriality on L_4, and the strong territoriality decreases the dispersion time much more (45%) compared to the L_3 case.

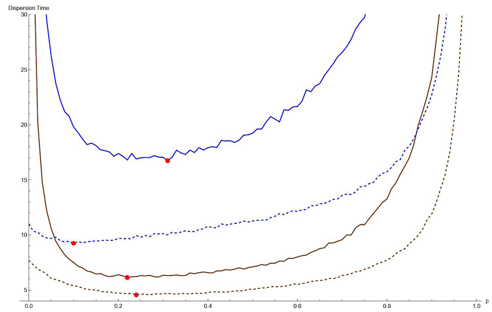

Figure 7: Dispersion time on L_3 (brown) and L_4 (blue) with (dashed) and without (solid) strong territoriality.

7.2 The case of C_3 and C_4

Like the line graph, the strong territorial behaviour significantly decreases the mean dispersion time on C_3 and C_4. For C_3, as shown in Figure 8, without territoriality, the minimum dispersion time is 4.41 with optimal $p = 0.26$ and with territoriality, the minimum dispersion time is 3.17 with $p = 0.38$, which is 28% less than the one without territoriality. Moreover, the effect becomes stronger when the network becomes more complex. On C_4, without territoriality, the minimum dispersion time is 11.68 with optimal $p = 0.27$, and with territoriality, the minimum dispersion time is 5.40 with $p = 0.34$, which is 54% less than the one without territoriality.

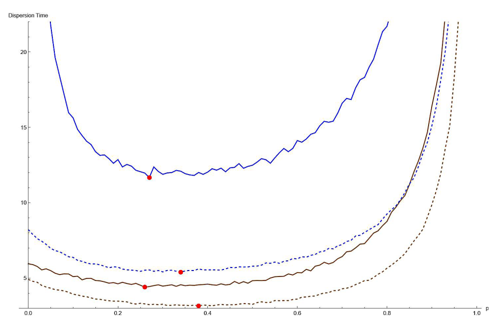

Figure 8: Dispersion time on C_3 (brown) and C_4 (blue) with (dashed) and without (solid) strong territoriality.

8 CONCLUSIONS

This article introduced the Social Distancing Problem on a connected graph, where agents have a common goal to have all their pairwise distances be at least a given number D. While different motions and information could be given to the agents for this problem, we give them only local knowledge of the graph and no knowledge of locations of other agents. So they know only the degree of their current node and lack memory. These assumptions limit the motions of the agents to Lazy Random Walks. We showed how to optimize their common laziness value p to achieve social distancing in the least expected number of steps. We considered various graphs and both exact and simulated methods. In some cases, the optimal motion was a random walk ($p = 0$) or a loop-random walk (choosing their current node with the same probability as each adjacent one). While we mostly consider the common-interest team version of the problem, we also studied cases where agents had individual selfish motives - we showed that in some cases, no symmetric equilibrium exists. Finally, we showed the territoriality decreases dispersion time.

We expect this area of research to be enlarged to other assumptions:

- Agents know the locations of some or all of the other agents.
- Agents have some memory.
- Agents know the whole graph.
- Agents can gain 'territoriality over a node'.

In this first paper on social distancing, we have restricted ourselves to considering only some simple classes of the graph and small sizes. It is to be hoped that further research in this area will find new and stronger methods able to study general graphs.

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